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AVERAGING FORMULA FOR NIELSEN NUMBERS

JONG BUM LEE AND KYUNG BAI LEE

ABSTRACT. We prove that the averaging formula for Nielsen numbers holds for continuous maps on infra-solvmanifolds of type (R): Let M be such a manifold with holonomy group Ψ and let $f : M \to M$ be a continuous map. The averaging formula for Nielsen numbers

$$N(f) = \frac{1}{|\Psi|} \sum_{A\in\Psi} |\det(A_* - f_*)|$$

is proved. This is a workable formula for the difficult number $N(f)$.

1. INTRODUCTION

Let M be a closed manifold and let $f : M \to M$ be a continuous map. A point $x \in M$ is called a *fixed point* of f if $f(x) = x$. This paper is concerned with the fixed points of maps. The two numbers $L(f)$ and $N(f)$ are associated to a map f, which are most important in the fixed point theory. The Lefschetz number $L(f)$ of f is defined by

$$L(f) = \sum_k (-1)^k \operatorname{trace}\{(f_*)_k : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})\}.$$

To define the Nielsen number $N(f)$ of f, we decompose the fixed point set $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ of f in a certain way into its subsets, called *fixed point classes* of f. To each fixed point class $\mathbb{F}$, one can assign an integer $\mathrm{ind}(f, \mathbb{F})$. The fixed

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point class $\mathbb{F}$ is called *essential* if $\mathrm{ind}(f,\mathbb{F}) \neq 0$. The Nielsen number $N(f)$ of f is defined to be

$$N(f) = \text{ the number of essential fixed point classes of } f.$$

The Lefschetz number and Nielsen number are homotopy invariants. If $L(f) \neq 0$, then every map homotopic to f has a fixed point. The Nielsen number $N(f)$ has the property that every map homotopic to f has at least $N(f)$ fixed points. The Nielsen number gives more precise information concerning the existence of fixed points than the Lefschetz number, but its computation when compared with that of the Lefschetz number is in general much more difficult.

Therefore, there have been attempts to find some relations between these two numbers. In [3], Brooks, Brown, Pak and Taylor show that for a continuous map f on a torus, $|L(f)| = N(f)$. Anosov [1] extended this to nilmanifolds. However, such an equality does not hold on infra-nilmanifolds (see below) as shown in [1]: there is a continuous map f on the Klein bottle for which $N(f) \neq |L(f)|$. If M is an infra-nilmanifold, and f is homotopically periodic or more generally virtually unipotent, then it is known in [11, 16] that $L(f) = N(f)$.

The following averaging formula for Lefschetz numbers is well-known [7, Theorem III.2.12]:

$$L(f) = \frac{1}{[\pi : K]} \sum_{\bar{\alpha} \in \pi/K} L(\bar{\alpha}\bar{f}).$$

For Nielsen numbers, the averaging formula does not always hold. Instead, the following inequality holds [9, Theorem 3.1]:

$$N(f) \geq \frac{1}{[\pi : K]} \sum_{\bar{\alpha} \in \pi/K} N(\bar{\alpha}\bar{f}).$$

It is proved in [9] that if X is an infra-nilmanifold, the above inequality becomes the equality. Furthermore, in [12], we offer algebraic and practical computation formulas for the Nielsen and Lefschetz numbers of any continuous maps on infra-nilmanifolds in terms of the holonomies of the manifolds.

The purpose of this work is to obtain algebraic and practical computation formulas for the Nielsen and Lefschetz numbers of any continuous maps on an infra-solvmanifold of type (R). Therefore we extend main results of [9, 12] from infra-nilmanifolds to infra-solvmanifolds of type (R), and thereby the deviation from the equality $N(f) = |L(f)|$ can be measured by the holonomy of the manifold.

2. Infra-solvmanifolds

Let G be a Lie group, let $\mathrm{Aut}(G)$ the group of continuous automorphisms of G and let $\mathrm{Endo}(G)$ the group of continuous endomorphisms of G. The affine group of G is the semi-direct product $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ with the multiplication $(a, A)(b, B) = (aA(b), AB)$. It has a Lie group structure and acts on G by $(a, A)\cdot x = aA(x)$ for all $x \in G$. Suppose that G is connected and has a linear connection defined by left-invariant vector fields. It turns out that $\mathrm{Aff}(G)$ is the group of connection preserving diffeomorphisms of G.

We recall from [4, 6, 10] some definitions about solvable Lie groups and give some basic properties which are necessary for our discussion. A connected solvable Lie group S is called of *type* (NR) (for "no roots") [10] if the eigenvalues of $\mathrm{Ad}(x) : \mathfrak{S} \to \mathfrak{S}$ are always either equal to 1 or else they are not roots of unity. Solvable Lie groups of type (NR) were considered first in [10]. A connected solvable Lie group S is called of *type* (R) (or *completely solvable*) if $\mathrm{ad}(X) : \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for each $X \in \mathfrak{S}$. A connected solvable Lie group S is called of *type* (E) (or *exponential*) if $\exp : \mathfrak{S} \to S$ is surjective. Some important properties of such groups related to our paper are listed below. See [4, 6] for more details.

(1) Abelian $\Longrightarrow$ Nilpotent $\Longrightarrow$ type (R) $\Longrightarrow$ type (E) $\Longrightarrow$ type (NR).
(2) (Rigidity of Lattices) Let S and S' be connected and simply connected solvable Lie groups of type (R), and let Γ be a lattice of S. Then any homomorphism from Γ to S' extends uniquely to a homomorphism of S to S'.

Let S be a connected and simply connected solvable Lie group and H be a closed subgroup of S. The coset space $H\backslash S$ is called a *solvmanifold*. We shall deal with *compact* solvmanifolds only. Let π be the fundamental group of the solvmanifold $M = H\backslash S$. Then $\pi = H/H_0$. Such a group is known to be a *Mostow-Wang* group or a *strongly torsion-free solvable* group, i.e., π contains a finitely generated torsion-free, nilpotent normal subgroup with torsion-free abelian quotient group of finite rank.

A discrete subgroup Γ of S is a *lattice* of S if $\Gamma\backslash S$ is compact, and in this case, we say that $\Gamma\backslash S$ is a *special* solvmanifold. Let $\pi \subset \mathrm{Aff}(S)$ be a torsion-free finite extension of Γ. Then π acts freely on S, and the manifold $\pi\backslash S$ is called an *infra-solvmanifold*. The group $\Psi = \pi/\Gamma$ is the holonomy group of π or $\pi\backslash S$. It sits naturally in $\mathrm{Aut}(S)$. Thus *every infra-solvmanifold finitely covers a special solvmanifold*. An infra-solvmanifold $M = \pi\backslash S$ is of type (R) if S is of type (R).

First we generalize Lemma 3.1 of [12], in which the existence of a fully invariant subgroup of finite index in an almost Bieberbach group is proved. The proof consists merely of straightforward adaptation of that of [12, Lemma 3.1] to this more general, but very analogous situation.

Lemma 2.1. *Let S, S' be connected and simply connected solvable Lie groups, and let $\pi, \pi' \subset \mathrm{Aff}(S)$ be finite extensions of lattices Γ, Γ' of S, S', respectively. Then there exist fully invariant subgroups $\Lambda \subset \Gamma, \Lambda' \subset \Gamma'$ of π, π', respectively, which are of finite index, so that any homomorphism $\theta : \pi \to \pi'$ maps Λ into Λ'.*

Next we state the following, which generalizes [14, Theorem 1.1] from almost crystallographic groups to finite extensions of lattices of a simply connected solvable Lie group of type (R), and [15, Theorem 3.1] from isomorphisms to homomorphisms. The crucial point is our Lemma 2.1 and the rigidity of lattices of simply connected solvable Lie groups of type (R). Then we just follow the argument of [14, Theorem 1.1] or [15, Theorem 3.1].

Theorem 2.2. *Let S be a connected and simply connected solvable Lie group of type* (R). *Let $\pi, \pi' \subset \mathrm{Aff}(S)$ be finite extensions of lattices of S. Then any homomorphism $\theta : \pi \to \pi'$ is semi-conjugate by an "affine map". That is, for any homomorphism $\theta : \pi \to \pi'$, there exist $d \in S$ and a homomorphism $D : S \to S$ such that $\theta(\alpha) \circ (d, D) = (d, D) \circ \alpha$, or the following diagram is commutative*

$$\begin{array}{ccc} S & \xrightarrow{(d,D)} & S \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \theta(\alpha)} \\ S & \xrightarrow{(d,D)} & S \end{array}$$

for all $\alpha \in \pi$.

3. Linearization of maps on solvmanifolds

In [13] (cf. [16] also), we exploited the fundamental group structure of solvmanifolds and produced, using Seifert fiber space constructions, a fibration structure on the solvmanifold over a torus, with a nilmanifold as a base. Let $M = \pi \backslash S$ be a solvmanifold. Then there is an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \mathbb{Z}^s \longrightarrow 1,$$

where

- Γ is a fully invariant subgroup of π, and
- Γ is a finitely generated torsion-free nilpotent group.

Let G be the Malćev completion of Γ. Doing the Seifert construction, one obtains a Seifert fibering with typical fiber the nilmanifold $\Gamma\backslash G$. The π action on $G \times \mathbb{R}^s$ is properly discontinuous, and free since π is torsion-free. Hence the Seifert fiber space $\pi\backslash(G\times\mathbb{R}^s)$ is a closed smooth manifold. This manifold has a bundle structure over the torus $\mathbb{Z}^s\backslash\mathbb{R}^s$ with fiber the nilmanifold $\Gamma\backslash G$:

$$\Gamma\backslash G \longrightarrow M' = \pi\backslash(G \times \mathbb{R}^s) \xrightarrow{q} \mathbb{Z}^s\backslash\mathbb{R}^s = T^s.$$

Note also that the given solvmanifold M is homotopic to the Seifert fiber space M'. Let $\alpha : M \to M'$ be a homotopy equivalence with a homotopy inverse β. Let $f : M \to M$. Consider $f' = \alpha \circ f \circ \beta : M' \to M'$. The homotopy commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \uparrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ M' & \xrightarrow{f'} & M' \end{array}$$

implies that $N(f) = N(f')$. Thus we may assume that M is the total space of the bundle $\Gamma\backslash G \to M \xrightarrow{q} T^s$. That is, $M = \pi\backslash(G \times \mathbb{R}^s)$, and $f : M \to M$.

Consider a lifting $\tilde{f} : G \times \mathbb{R}^s \to G \times \mathbb{R}^s$ of $f : M \to M$. Then it induces a homomorphism $\varphi : \pi \to \pi$ defined by $\varphi(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha$ for all $\alpha \in \pi$. Since Γ is a fully invariant subgroup of π, φ induces a homomorphism $\varphi' = \varphi|_\Gamma : \Gamma \to \Gamma$, and in turn induces a homomorphism $\bar{\varphi} : \mathbb{Z}^s \to \mathbb{Z}^s$ so that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \xrightarrow{\psi} & \mathbb{Z}^s & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar{\varphi}} & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \xrightarrow{\psi} & \mathbb{Z}^s & \longrightarrow & 1 \end{array}$$

Now $\bar{\varphi} : \mathbb{Z}^s \to \mathbb{Z}^s$ extends uniquely to a homomorphism $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$, which induces a map $\phi_{\bar{F}} : T^s \to T^s$ so that the induced homomorphism on the group of covering transformations, $\mathbb{Z}^s$, is exactly $\bar{\varphi}$. Since $\bar{\varphi}\psi = \psi\varphi$, we have a homotopy $h_t : q \circ f \simeq \phi_{\bar{F}} \circ q$. By the Homotopy Lifting Property of the fibration $M \to T^s$, there exists a lifting homotopy $H_t : M \to M$ of h_t such that $H_0 = f$. Let $f' = H_1$. Then $q \circ f' = \phi_{\bar{F}} \circ q$, and f' is fiber-preserving and homotopic to f. Moreover, $f' : M \to M$ induces the map $\phi_{\bar{F}} : T^s \to T^s$ which is a homomorphism.

We recall the definition of the linearization of the solvmanifold [10]. There exists a finite descending central series of Γ:

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_c \supset \cdots$$

such that

- each $\Gamma_i := \Gamma \cap \gamma_i(G)$ is a fully invariant subgroup of Γ,
- each $\Lambda_i := \Gamma_{i-1}/\Gamma_i$ is torsion-free abelian,
- Γ acts trivially on each Λ_i, and
- there is a well-defined action of $\mathbb{Z}^s$ on each Λ_i. We denote its action homomorphism by $A_i : \mathbb{Z}^s \to \mathrm{Aut}(\Lambda_i)$.

Then the collection $\{\Lambda_i, A_i\}$ is called the linearization of the solvmanifold M.

Recall that $\varphi' : \Gamma \to \Gamma$ extends uniquely to a homomorphism $F' : G \to G$. Then F' induces homomorphisms $\gamma_i(G) \to \gamma_i(G)$ and then in turn induces homomorphisms

$$\varphi_i' = \varphi'|_\Gamma : \Gamma_i = \Gamma \cap \gamma_i(G) \longrightarrow \Gamma_i.$$

Therefore, there are induced homomorphisms

$$\hat{\varphi}_i' : \Lambda_i \longrightarrow \Lambda_i.$$

Furthermore the above commutative diagram produces the following equality: for each i and $\lambda \in \mathbb{Z}^s$,

$$\hat{\varphi}_i' \circ A_i(\lambda) = A_i(\bar{\varphi}(\lambda)) \circ \hat{\varphi}_i'.$$

We shall say that the *linearization* of the map $f : M \to M$ is the collection of homomorphisms $\{\hat{\varphi}_i', \bar{\varphi}\}$, or simply, the pair $(F', \bar{F})$ of the homomorphisms $F' : G \to G$ and $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$. Notice that the differential of F', $F_*' : \mathfrak{G} \to \mathfrak{G}$, can be expressed as a matrix of the form

$$\begin{bmatrix} \hat{\varphi}_1' & * & \dots & * \\ 0 & \hat{\varphi}_2' & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{\varphi}_c' \end{bmatrix}$$

by choosing a suitable basis of $\mathfrak{G}$.

Therefore, the main result of [10] can be re-stated as follows:

Theorem 3.1 ([10, Theorem 3.1]). *If $f : M \to M$ is a self-map on the solvmanifold of type* (NR) *with linearization $(F', \bar{F})$, then*

$$L(f) = \det(I - \bar{F}_*) \det(I - F_*'), \quad N(f) = |L(f)|.$$

Corollary 3.2. *Let $M = \Gamma\backslash S$ be a special solvmanifold of type* (NR)*. If a homomorphism $D : S \to S$ induces a map $\phi_D : M \to M$, then $L(\phi_D) = \det(I - D_*)$ and $N(\phi_D) = |L(\phi_D)|$.*

The following slightly generalizes [12, Lemma 3.2] from simply connected nilpotent Lie groups to simply connected solvable Lie groups.

Lemma 3.3. *Let S be a connected and simply connected solvable Lie group. For any $g \in S$ and $D \in \mathrm{Endo}(S)$,*

$$\det(\mathrm{Ad}(g) - D_*) = \det(I - D_*).$$

4. Averaging formula for Nielsen numbers

To state and prove the main result, we shall briefly explain some necessary facts about the mod K Nielsen fixed point theory. Let $f : X \to X$ be a self-map on a compact connected space X, and let K be a normal subgroup of $\pi = \pi_1(X)$ of finite index. Suppose $f_*(K) \subset K$. For the fixed liftings $\tilde{f} : \tilde{X} \to \tilde{X}$ and $\bar{f} : K\backslash\tilde{X} \to K\backslash\tilde{X}$ of f, we have homomorphisms

$$\bar{\varphi} : \pi/K \to \pi/K \quad \text{defined by } \bar{f}\bar{\alpha} = \bar{\varphi}(\bar{\alpha})\bar{f},$$
$$\varphi : \pi \to \pi \quad \text{defined by } \tilde{f}\alpha = \varphi(\alpha)\tilde{f},$$

so that $\varphi' = \varphi|_K : K \to K$ and the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\ 1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \end{array}$$

Let $p : \tilde{X} \to X$, $p' : \tilde{X} \to K\backslash X$ be covering maps. The fixed point classes of f are the subsets $p(\mathrm{Fix}(\alpha\tilde{f}))$ $(\alpha \in \pi)$ of the fixed point set $\mathrm{Fix}(f)$ of f. For each $\alpha \in \pi$, the fixed point classes of $\bar{\alpha}\bar{f}$ are the subsets $p'(\mathrm{Fix}(k\alpha\tilde{f}))$ $(k \in K)$ of the fixed point set $\mathrm{Fix}(\bar{\alpha}\bar{f})$ of $\bar{\alpha}\bar{f}$.

We denote the subgroup of π fixed by a homomorphism $\psi : \pi \longrightarrow \pi$ by

$$\mathrm{fix}(\psi) = \{\alpha \in \pi \mid \psi(\alpha) = \alpha\}.$$

Then the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau_\alpha\varphi'} & & \downarrow{\scriptstyle \tau_\alpha\varphi} & & \downarrow{\scriptstyle \tau_{\bar{\alpha}}\bar{\varphi}} & & \\ 1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \end{array}$$

is commutative, and the following sequence of groups

$$1 \to \mathrm{fix}(\tau_\alpha\varphi') \xrightarrow{i} \mathrm{fix}(\tau_\alpha\varphi) \xrightarrow{q} \mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi})$$

is exact. With the above notation, we have

Theorem 4.1 ([9, Theorem 3.1]). *Let $f : X \to X$ be a self-map on a compact connected space X, and let K be a normal subgroup of the fundamental group π of X with finite index such that $f_*(K) \subset K$. Then*

$$N(f) \geq \frac{1}{[\pi : K]} \sum_{\bar{\alpha} \in \pi/K} N(\bar{\alpha}\bar{f}),$$

and equality holds if and only if for each $k \in K$ and $\alpha \in \Pi$ with $p(\mathrm{Fix}(k\alpha\tilde{f}))$ an essential fixed point class, $|q(\mathrm{fix}(\tau_{k\alpha}\varphi))| = 1$.

We will show that the equality in the above theorem holds for infra-solvmanifolds of type (R). Thus we generalize the averaging formula for Nielsen numbers for continuous maps on infra-nilmanifolds [9, Theorem 3.5] to infra-solvmanifolds of type (R). With Theorem 2.2, our proof requires a straightforward adaptation of the proof of [9, Theorem 3.5].

Theorem 4.2. *Let M be an infra-solvmanifold of type* (R) *and $f : M \to M$ be any self map. Suppose that N is a regular covering of M which is a solvmanifold with fundamental group K. Assume that $f_*(K) \subset K$. Then*

$$N(f) = \frac{1}{[\pi : K]} \sum N(\bar{f}),$$

where the sum ranges over all the liftings $\bar{f}$ of f onto N. In particular, $N(f) \geq |L(f)|$.

We now explain how a continuous map $f : M \to M$ on the infra-solvmanifold $M = \pi\backslash S$ of type (R) induces an endomorphism of the Lie algebra, $f_* : \mathfrak{S} \to \mathfrak{S}$, naturally. The continuous map $f : M \to M$ induces a homomorphism $\varphi : \pi \to \pi$. Let Λ be a fully invariant subgroup of π in Lemma 2.1. Note that Λ is a lattice of S. Then, the induced homomorphism $\varphi : \pi \to \pi$ restricts to a homomorphism $\varphi' = \varphi|_\Lambda : \Lambda \to \Lambda$, which extends to an endomorphism of the Lie group S in a unique way. See [6, Theorem 2.2]. The differential of this map is an endomorphism of the Lie algebra, $f_* : \mathfrak{S} \to \mathfrak{S}$.

In all, we can prove our main result which computes the Lefschetz number $L(f)$ and the Nielsen number $N(f)$ of any continuous map f on an infra-solvmanifold M of type (R) in terms of the *holonomy* of the manifold. Thus we generalize the algebraic computation formula for Nielsen numbers for continuous maps on infra-nilmanifolds [12, Theorem 3.4] to infra-solvmanifolds of type (R). Our proof requires a straightforward adaptation of the proof of [12, Theorem 3.4]. Namely,

Theorem 4.3. *Let $f : M \to M$ be any continuous map on an infra-solvmanifold M of type* (R) *with the holonomy group Ψ. Then*

$$L(f) = \frac{1}{|\Psi|} \sum_{A\in\Psi} \frac{\det(A_* - f_*)}{\det A_*},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A\in\Psi} |\det(A_* - f_*)|.$$

Example 4.4. The solvable Lie group Sol is one of the eight geometries that one considers in the study of 3-manifolds [18]. One can describe Sol as a semi-direct product $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the map

$$\varphi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Its Lie algebra $\mathfrak{sol}$ is given as $\mathfrak{sol} = \mathbb{R}^2 \rtimes_\sigma \mathbb{R}$ where

$$\sigma(t) = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}.$$

The Lie group Sol can be embedded into Aff(3) as

$$\begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where x, y and t are real numbers, and hence its Lie algebra $\mathfrak{sol}$ is isomorphic to the algebra of matrices

$$\begin{bmatrix} t & 0 & 0 & a \\ 0 & -t & 0 & b \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since Sol is of type (R), it is of type (NR). We denote the general element of Sol by $\{x, y, t\}$. Let Γ be the subgroup of Sol which is generated by

$$\{\tfrac{1}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}, 0\}, \quad \{\tfrac{\sqrt{5}-1}{2\sqrt{5}}, \tfrac{\sqrt{5}+1}{2\sqrt{5}}, 0\}, \quad \{0, 0, \ln \tfrac{3+\sqrt{5}}{2}\}.$$

Then Γ is isomorphic to the group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ where

$$\phi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is an element of $\mathrm{SL}(2, \mathbb{Z})$ with eigenvalues $\frac{3\pm\sqrt{5}}{2}$, and in fact Γ is a lattice of Sol.

Let $a=\{0,0,\frac{1}{2}\ln\frac{3+\sqrt{5}}{2}\}\in\mathrm{Sol}$ and $A:\mathrm{Sol}\to\mathrm{Sol}$ be the automorphism of Sol given by

$$A(\{x,y,t\})=\{-x,-y,t\}.$$

Then A has period 2, and $(a,A)^2=(\{0,0,\ln\frac{3+\sqrt{5}}{2}\},I)\in\mathrm{Sol}\rtimes\mathrm{Aut(Sol)}$, where I is the identity automorphism of Sol. The subgroup

$$\pi=\langle\Gamma,(a,A)\rangle\subset\mathrm{Sol}\rtimes\mathrm{Aut(Sol)}$$

generated by the lattice Γ and the element (a,A) is discrete and torsion free, and Γ is a normal subgroup of π of index 2. Thus π is a torsion-free finite extension of the lattice Γ, and $\pi\backslash\mathrm{Sol}$ is an infra-solvmanifold, which has a double covering $\Gamma\backslash\mathrm{Sol}\to\pi\backslash\mathrm{Sol}$ by its holonomy group, $\Psi=\pi/\Gamma=\{1,A\}\cong\mathbb{Z}_2$.

Let $D:\mathrm{Sol}\to\mathrm{Sol}$ be the automorphism of Sol given by

$$D(\{x,y,t\})=\{my,mx,-t\}$$

where m is any nonzero integer (cf. [5, Proposition 2.3]). Then $DA=AD$ and the conjugation by $(\{0,0,0\},D)\in\mathrm{Sol}\rtimes\mathrm{Aut(Sol)}$ maps π into π (and Γ into Γ). Thus, the affine map $(\{0,0,0\},D):\mathrm{Sol}\to\mathrm{Sol}$ induces $\phi_D:\Gamma\backslash\mathrm{Sol}\to\Gamma\backslash\mathrm{Sol}$ and $\Phi_D:\pi\backslash\mathrm{Sol}\to\pi\backslash\mathrm{Sol}$ so that the following diagram is commutative:

$$\begin{array}{ccc}
(\pi,\mathrm{Sol}) & \xrightarrow{(\{0,0,0\},D)} & (\pi,\mathrm{Sol})\\
\downarrow & & \downarrow\\
(\pi/\Gamma,\Gamma\backslash\mathrm{Sol}) & \xrightarrow{\phi_D} & (\pi/\Gamma,\Gamma\backslash\mathrm{Sol})\\
\downarrow & & \downarrow\\
\pi\backslash\mathrm{Sol} & \xrightarrow{\Phi_D} & \pi\backslash\mathrm{Sol}
\end{array}$$

We take an ordered (linear) basis for the Lie algebra of Sol as follows:

$$\mathbf{e}_1=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathbf{e}_2=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathbf{e}_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}.$$

With respect to this basis, the differentials of A and D are

$$A_*=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\quad D_*=\begin{bmatrix}0&m&0\\m&0&0\\0&0&-1\end{bmatrix}.$$

Therefore by Theorem 4.3, the Lefschetz number and the Nielsen number of the map $\Phi_D : \pi\backslash\mathrm{Sol} \to \pi\backslash\mathrm{Sol}$ are:

$$\begin{aligned} L(\Phi_D) &= \frac{1}{2}\left(\frac{\det(I - D_*)}{\det(I)} + \frac{\det(A_* - D_*)}{\det(A_*)}\right) \\ &= \frac{1}{2}\big(2(1-m^2) + 0\big) = 1 - m^2, \end{aligned}$$

$$\begin{aligned} N(\Phi_D) &= \frac{1}{2}\left(|\det(I - D_*)| + |\det(A_* - D_*)|\right) \\ &= \frac{1}{2}\big(|2(1-m^2)| + |0|\big) = |1 - m^2|. \end{aligned}$$

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Department of Mathematics, Sogang University, Seoul 121-742, KOREA

E-mail address: jlee@sogang.ac.kr

Department of Mathematics, University of Oklahoma, Norman, OK 73019, U.S.A.

E-mail address: kblee@math.ou.edu