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A CAUCHY PROBLEM OF SINE-GORDON EQUATIONS WITH NON-HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

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Abstract. In this paper we will establish existence, uniqueness and continuous dependence on the data for the damped sine-Gordon equations with non-homogeneous Dirichlet boundary conditions in a weaker sense.

1. INTRODUCTION

Let Ω be an open bounded subset of R^n with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. In this paper we will study study existence, uniqueness and continuous dependence on the initial data for damped sine-Gordon equations with a non-homogeneous Dirichlet boundary condition:

$$(1.1) \qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \alpha \dfrac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y + hy = f \quad \text{in} \quad Q, \\ y = g \quad \text{on} \quad \Sigma, \\ y(0,x) = y_0(x) \quad \text{and} \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \quad \Omega, \end{cases}$$

where $\alpha, \beta, \gamma \in R, \beta > 0$ are physical constants, $h \in L^\infty(0,T; L^\infty(\Omega))$ is a multiplier function, f is a forcing function, g is a boundary forcing function and y_0, y_1 are initial values. This equation describes the dynamics of Josephson junction driven by a current source by taking into account of damping effect. Also the equation (1.1) with $g = h = 0$ is well known as a specific equation which causes chaotic behaviors in Bishop, Fesser and Lomdahl [2] and Levi [4]. Non-homogeneous boundary value problems for linear second order evolution equations are studied extensively in Lions and Magenes [6] and Dautray and Lions [3] through the method of transpositions. The linear theory has been completed by the above books, but the researches on nonlinear cases are not sufficient. For nonlinear studies, we can refer the books by Banks, Smith and Wang [1], Temam [9], Lions [5] and Taylor [8]. Further, many researches on nonlinear equations are focused on *standard* boundary conditions, i.e., the Dirichlet zero or the Neumann boundary conditions.

For the *standard* boundary value problems of (1.1) we have proved the existence and uniqueness of weak solutions in Ha and Nakagiri [7] in abstract evolution equation setting. However, the method used in [7] cannot be applied to the non-homogeneous Dirichlet boundary condition case. Therefore we utilize the method of *transposition*, or the *adjoint isomorphism* of equations, and we shall solve the case of non-homogeneous Dirichlet boundary conditions under weaker assumptions on

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the data than those in [7]. That is, it is our main purpose of this paper to establish a new well-posedness result for (1.1) with non-homogeneous Dirichlet boundary conditions, by using the method of *transposition* which is suitably set for our nonlinear case.

This paper is composed of two parts. In section 2 we explain the classical well-posedness result for the problem (1.1) with homogeneous boundary conditions. After applying the transposition method to damped linear equations, we establish the existence, uniqueness, continuous dependence and regularity of generalized solutions for the problem (1.1) in section 3.

2. HOMOGENEOUS BOUNDARY CONDITION

In this section we study the following initial and homogeneous boundary value problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \alpha \dfrac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y + hy = f \quad \text{in} \quad Q, \\ y = 0 \quad \text{on} \quad \Sigma, \\ y(0,x) = y_0(x) \quad \text{in} \quad \Omega \quad \text{and} \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $\alpha, \gamma \in R \equiv (-\infty, \infty)$, $\beta > 0$, Δ is a Laplacian, $h \in L^\infty(0,T; L^\infty(\Omega))$ is a multiplication function, f is a given forcing function, y_0, y_1 are initial values. We shall give the classical well-posedness result for (2.1) under the stronger assumptions on f and y_0, y_1.

In order to treat the problem (2.1) in variational formulation, we introduce two Hilbert spaces H and V by $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, respectively. We endow these spaces with the usual inner products and norms

$$(\psi, \phi) = \int_\Omega \psi(x)\phi(x)dx, \quad |\psi| = (\psi,\psi)^{1/2} \quad \text{for all} \quad \phi, \psi \in L^2(\Omega),$$

$$(\!(\psi, \phi)\!) = \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}\psi(x)\frac{\partial}{\partial x_i}\phi(x)dx, \quad \|\psi\| = (\!(\psi,\psi)\!)^{1/2} \quad \text{for all} \quad \phi, \psi \in H_0^1(\Omega).$$

Then the pair (V, H) is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and $V' = H^{-1}(\Omega)$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. By $\langle \cdot, \cdot \rangle$ we denote the dual pairing between V' and V. To use the variational formulation let us introduce the bilinear form

$$a(\phi, \varphi) = \int_\Omega \nabla\phi \cdot \nabla\varphi dx = (\!(\phi, \varphi)\!) \quad \text{for all} \quad \phi, \varphi \in V = H_0^1(\Omega).$$

Then this form is symmetric, bounded on $H_0^1(\Omega) \times H_0^1(\Omega)$ and coercive, i.e.,

$$a(\phi, \phi) \geq \|\phi\|^2 \quad \text{for all} \quad \phi \in H_0^1(\Omega).$$

Then we can define the bounded operator $A \in \mathcal{L}(V, V')$ by the relation $a(\phi, \psi) = \langle A\phi, \psi \rangle$ and the equation (2.1) is reduced to a damped second order equation in H of the form

$$\begin{cases} \dfrac{d^2 y}{dt^2} + \alpha \dfrac{dy}{dt} + \beta Ay + \gamma \sin y + hy = f \quad \text{in} \quad (0,T), \\ y(0) = y_0 \in V, \quad \dfrac{dy}{dt}(0) = y_1 \in H. \end{cases} \tag{2.2}$$

The operator A in (2.2) is an isomorphism from V onto V' and it is also considered as a self-adjoint unbounded operator in H with dense domain $\mathcal{D}(A)$ in V and in H,

$$\mathcal{D}(A) = \{\phi \in V : \ A\phi \in H\}.$$

By $\mathcal{D}'(0,T;X)$ we denote the space of distributions from $\mathcal{D}(\Omega)$ into X, where X is a Hilbert space. If $X = R$, $\mathcal{D}'(0,T;X)$ is simply denoted by $\mathcal{D}'(0,T)$. We shall write $g' = \frac{dg}{dt}, g'' = \frac{d^2g}{dt^2}$, of which derivatives are taken in the distribution sense $\mathcal{D}'(0,T;V)$. We define the Hilbert space of solutions $W(0,T)$ by

$$W(0,T) = \{g | g \in L^2(0,T;V), g' \in L^2(0,T;H), g'' \in L^2(0,T;V')\}$$

with the scalar product defined by

$$(f,g)_W = \int_0^T (\!(f,g)\!) dt + \int_0^T (f',g')dt + \int_0^T (\!(f'',g'')\!)_{V'} dt,$$

where $(\!(\cdot,\cdot)\!)_{V'}$ denotes the inner product on V'.

Definition 2.1. A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\begin{aligned} &\langle y''(\cdot), \phi\rangle + (\alpha y'(\cdot), \phi) + (\!(\beta y(\cdot), \phi)\!) + (\gamma \sin y(\cdot), \phi) + (h(\cdot)y(\cdot), \phi) = (f(\cdot), \phi) \\ &\qquad\qquad\qquad \text{for all } \ \phi \in V \ \text{ in the sense of } \ \mathcal{D}'(0,T), \\ &y(0) = y_0, \quad y'(0) = y_1. \end{aligned}$$

The following theorem on the existence, uniqueness and regularity of solutions for (2.1) is proved in Ha and Nakagiri [7].

Theorem 2.2. Let $\alpha, \gamma \in R, \beta > 0$, $h \in L^\infty(0,T;L^\infty(\Omega))$ and $f,\ y_0,\ y_1$ be given satisfying

$$f \in L^2(0,T;L^2(\Omega)), \quad y_0 \in H_0^1(\Omega), \quad y_1 \in L^2(\Omega).$$

Then the problem (2.1) has a unique weak solution y in $W(0,T)$ and y has the regularity

$$y \in C([0,T];H_0^1(\Omega)), \quad y' \in C([0,T];L^2(\Omega)).$$

Furthermore we have the estimates:

$$|y'(t)|^2 + \|y(t)\|^2 \le c(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}), \ \forall t \in [0,T], \tag{2.3}$$

where c is a constant depending only on α, β, γ and $\|h\|_{L^\infty(0,T;L^\infty(\Omega))}$.

Remark 1. Theorem 2.2 holds true for the problem (2.1) in which the nonlinear term $\sin y$ is replaced by $\sin(y_L + y)$ for some fixed $y_L \in L^2(0,T;L^2(\Omega))$.

3. NON-HOMOGENEOUS BOUNDARY CONDITION

We consider the following non-homogeneous initial-boundary value problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \alpha \dfrac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y + hy = f \ \text{ in } \ Q, \\ y = g \ \text{ on } \ \Sigma, \\ y(x,0) = y_0(x), \quad \dfrac{\partial y}{\partial t}(x,0) = y_1(x) \ \text{ in } \ \Omega. \end{cases} \tag{3.1}$$

We want to solve the problem (3.1) under weaker conditions on the data f, g, y_0, y_1 than those given in Section 2.

For this, we shall introduce the definition of weak integral solutions for (3.1) which is suggested by the method of *transposition* for linear equations which is studied extensively in Lions and Magenes [6].

First we recall the transposition method for linear equations. Let $\tilde{h} \in L^\infty(0,T;L^\infty(\Omega))$ be fixed. By Theorem 2.2 with $\gamma = 0$, for each $\tilde{f} \in L^2(0,T;H)$ there exists an unique weak solution $\phi = \phi(\tilde{f}) \in W(0,T)$ of the linear problem

$$\tag{3.2} \left\{ \begin{array}{l} \phi'' - \alpha\phi' + \beta A\phi + \tilde{h}\phi = \tilde{f} \quad \text{in} \quad (0,T), \\ \phi(T) = \phi'(T) = 0. \end{array} \right.$$

Indeed it is easily checked if we consider the time reversion, i.e., $\psi(t) = \phi(T-t)$.

Let $X_{\tilde{h}}$ be the set of all functions ϕ satisfying (3.2) for each $\tilde{f} \in L^2(0,T;H)$. We also give an inner product on $X_{\tilde{h}}$ by

$$(\!(\phi(\tilde{f}), \phi(\tilde{g}))\!)_{X_h} = (\!(\tilde{f}, \tilde{g})\!)_{L^2(0,T;H)},$$

where $\phi(\tilde{f})$ denotes the weak solution to (3.2) for a given $\tilde{f}$. Then it is easily checked that $(X_{\tilde{h}}, (\!(\cdot,\cdot)\!)_{X_{\tilde{h}}})$ is a Hilbert space. Hence the mapping $\mathcal{L}_{\tilde{h}} : X_{\tilde{h}} \to L^2(0,T;H)$ defined by

$$\phi \to \phi'' - \alpha\phi' + \beta A\phi + \tilde{h}\phi$$

is an isomorphism. Since $X_{\tilde{h}} \subset W(0,T)$ as a set, we have by (2.3) that

$$\tag{3.3} \|\mathcal{L}_{\tilde{h}}^{-1}\tilde{f}\|_{L^2(0,T;V)} + \|\frac{d}{dt}\mathcal{L}_{\tilde{h}}^{-1}\tilde{f}\|_{L^2(0,T;H)} \le c\|\tilde{f}\|_{L^2(0,T;H)}, \quad \exists c > 0,$$

where c depends on $\|\tilde{h}\|_{L^\infty(0,T;L^\infty(\Omega))}$.

For simplicity of notations, we denote $X = X_h$ and $\mathcal{L} = \mathcal{L}_h$, where h is the function given in the equation (2.1). Note that $X = X_{\tilde{h}}$ in $W(0,T)$ for any $\tilde{h} \in L^\infty(0,T;L^\infty(\Omega))$.

The following theorem is now immediate from the isomorphism $\phi \in X \to \phi'' - \alpha\phi' + \beta A\phi + h\phi \in L^2(0,T;H)$.

Theorem 3.1. Let l be a bounded linear functional on X. Then there exists a unique solution $y \in L^2(0,T;H)$ such that

$$\tag{3.4} \int_0^T (y, \phi'' - \alpha\phi' + \beta A\phi + h\phi)dt = l(\phi), \quad \forall \phi \in X.$$

Now we give the definition of a weak integral solution of (3.1).

Definition 3.2. We assume $h \in L^\infty(0,T;L^\infty(\Omega))$, $f \in L^1(0,T;V')$, $g \in L^1(0,T;H^{\frac{1}{2}}(\Gamma))$ and $y_0 \in H = L^2(\Omega)$, $y_1 \in V' = H^{-1}(\Omega)$. A function y is said to be a weak integral solution of (3.1) if $y \in L^2(0,T;H)$ and y satisfies

$$\tag{3.5} \begin{aligned} \int_0^T (y, \phi'' - \alpha\phi' + \beta A\phi + h\phi)dt &= \int_0^T \langle f, \phi\rangle dt - \gamma\int_0^T (\sin y, \phi)dt \\ &+ (\alpha y_0, \phi(0)) + \langle y_1, \phi(0)\rangle - (y_0, \phi'(0)) - \int_0^T \langle g, \beta\frac{\partial\phi}{\partial\mathbf{n}}\rangle_\Gamma dt, \quad \forall \phi \in X, \end{aligned}$$

where $\langle\psi,\phi\rangle_\Gamma$ is the duality pairing between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$.

Note that $\frac{\partial\phi}{\partial\mathbf{n}} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$ by $\phi \in X \subset W(0,T)$. The sum of all terms of (3.5) excepting the sine term is shown to be a bounded linear functional on X. Hence, if $\gamma = 0$ in Definition 3.2, then the weak integral solution coincides with the solution by the transposition method in Theorem 3.1.

We look for the weak integral solution of (3.1) as the sum $y_L + z$ of the following two linear and nonlinear problems:

$$\begin{cases} \dfrac{\partial^2 y_L}{\partial t^2} + \alpha \dfrac{\partial y_L}{\partial t} - \beta \Delta y_L + h y_L = f \ \text{ in } \ Q, \\ y_L = g \ \text{ on } \ \Sigma, \\ y_L(x,0) = y_0(x), \ \ \dfrac{\partial y_L}{\partial t}(x,0) = y_1(x) \ \text{ in } \ \Omega. \end{cases} \tag{3.6}$$

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} + \alpha \dfrac{\partial z}{\partial t} - \beta \Delta z + \gamma \sin(y_L + z) + hz = 0 \ \text{ in } \ Q, \\ z = 0 \ \text{ on } \ \Sigma, \\ z(x,0) = 0, \ \ \dfrac{\partial z}{\partial t}(x,0) = 0 \ \text{ in } \ \Omega. \end{cases} \tag{3.7}$$

Theorem 3.3. Let $\alpha, \gamma \in R$, $\beta > 0$, $h \in L^\infty(0,T; L^\infty(\Omega))$ and the data $f, g, \ y_0, \ y_1$ be given satisfying

$$f \in L^1(0,T; H^{-1}(\Omega)), \quad g \in L^1(0,T; H^{\frac{1}{2}}(\Gamma)), \quad y_0 \in L^2(\Omega), \quad y_1 \in H^{-1}(\Omega).$$

Then the equation (3.6) has a unique weak integral solution $y_L \in L^2(0,T; L^2(\Omega))$.

Now we are ready to state our main theorem.

Theorem 3.4. Under the assumptions in Theorem 3.3, there exists a unique weak integral solution $y \in L^2(0,T; L^2(\Omega))$ of (3.1). In addition the solution y is continuously depending on the initial data y_0, y_1 and forcing and boundary functions f, g.

Remark 2. We can easily extend Theorem 3.4 to general equations in which α and $\beta\Delta$ are replaced by the differential operators depending on (t,x). Also we can extend the equations having bounded C^1-class nonlinear function terms.

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