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BRIDGES BETWEEN HYPONORMAL AND SUBNORMAL OPERATORS

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ABSTRACT. There are several bridges to detect the gap between subnormal and hyponormal operators on a Hilbert space. In this article we first discuss a bridge through the classes of (strongly) k-hyponormal operators. Another bridge can be obtained through the classes of weakly k-hyponormal operators. To detect those bridges, we consider weighted shifts and their backward extensions with variables. In particular, we discuss quadratically hyponormal and cubically hyponormal weighted shifts and introduce some related open problems.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. It is well known that normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal, with converses false. An n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, T is k-*hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. The n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_n T_n$ is hyponormal for every $\lambda_i \in \mathbf{C},\ i = 1, \cdots, n$, where $\mathbf{C}$ is the set of complex numbers. An operator T is *weakly* k-*hyponormal* if $(T, T^2, \cdots, T^k)$ is weakly hyponormal. It is well-known that subnormal $\Rightarrow$ k-hyponormal $\Rightarrow$ weakly k-hyponormal, for every $k \geq 1$; to study the converse, unilateral weighted shifts were considered in [2], [4], [5], [13], and [14]. In an attempt to bridge the gap between hyponormality and subnormality, the classes of k-hyponormal and weakly k-hyponormal operators were introduced and studied in [1], [2], [3], [4], [5], [7], [11], [12] and [8].

In this article we review k-hyponormal, weakly k-hyponormal weighted shifts and their related open problems. In Section 2, we review the structure of quadratically hyponormal weighted shifts. First we characterize the quadratical hyponormality of weighted shifts and show that the weighted shift $W_{\alpha(x)}$ is quadratically hyponormal if and only if it is positively quadratically hyponormal, where $\alpha(x) : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $0 < x \leq y$. Also we discuss the quadratically hyponormal weighted shifts with two equal weights and show that the region $\mathcal{R} :=$

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$\{(x,y)|W_\alpha$ is quadratically hyponormal$\}$ is a closed convex set with nonempty interior. In Section 3 we review cubically hyponormal weighted shifts and obtain an example which is cubically hyponormal but not 2-hyponormal. In addition, we introduce several related open problems in each sections.

We first recall [4] that a weighted shift W_α is said to be *recursively generated* if there exist $i \geq 1$ and $\Psi = (\Psi_0, \cdots, \Psi_{i-1}) \in \mathbf{C}^i$ such that

$$\gamma_n = \Psi_{i-1}\gamma_{n-1} + \cdots + \Psi_0\gamma_{n-i} \qquad (n \geq i), \tag{1}$$

where $\gamma_n\,(n \geq 0)$ is the moment of W_α, i.e., $\gamma_0 := 1, \gamma_{n+1} := \alpha_n^2\gamma_n \;\; (n \geq 0)$. Furthermore, (1) is equivalent to

$$\alpha_n^2 = \Psi_{i-1} + \frac{\Psi_{i-2}}{\alpha_{n-1}^2} + \cdots + \frac{\Psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-i+1}^2} \qquad (n \geq i).$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_{2k}$ $(k \geq 0)$, there is a canonical procedure to generate a sequence (denote $\widehat{\alpha}$) in such a way that $W_{\widehat{\alpha}}$ is a recursively generated shift having α as an initial segment of weights (cf. [4, p. 219]). We now review this procedure in a special case. Given $\alpha : \alpha_0, \alpha_1, \alpha_2$ $(0 < \alpha_0 < \alpha_1 < \alpha_2)$, let

$$\mathbf{v}_0 := \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \quad \mathbf{v}_1 := \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad \mathbf{v}_2 := \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

The vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ are linearly independent in $\mathbf{R}^2$, so there exists a unique $\Psi = (\Psi_0, \Psi_1) \in \mathbf{R}^2$ such that $\mathbf{v}_2 = \Psi_0\mathbf{v}_0 + \Psi_1\mathbf{v}_1$. In fact,

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

Let $\widehat{\gamma}_n := \gamma_n$ $(0 \leq n \leq 1)$ and let $\widehat{\gamma}_n := \Psi_1\widehat{\gamma}_{n-1} + \Psi_0\widehat{\gamma}_{n-2}$ $(n \geq 2)$. Since $\widehat{\gamma}_n > 0\,(n \geq 0)$ (cf. [4]), we define

$$\widehat{\alpha}_n := \left(\frac{\widehat{\gamma}_{n+1}}{\widehat{\gamma}_n}\right)^{\frac{1}{2}} \qquad (n \geq 0)$$

(so that $\widehat{\alpha}_n = \alpha_n$ for $0 \leq n \leq 2$). Hence we obtain the coefficients of a recursively generated weighted shift, and

$$\widehat{\alpha}_n^2 = \Psi_1 + \frac{\Psi_0}{\widehat{\alpha}_{n-1}^2} \qquad (n \geq 1).$$

Some of the proof and examples here were obtained through computer experiments using the software tool *Mathematica* [15].

2. Quadratically hyponormal Weighted Shifts

The weakly k-hyponormality provides a nice bridge connecting in discrete between hyponormal and subnormal operators. In particular, the case $n = 2$ often refered to as quadratically hyponormal operators. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is *quadratically hyponormal* if $T + sT^2$ is hyponormal for every $s \in \mathbf{C}$. Let $\{e_k\}_{k=0}^\infty$ be an orthonormal basis for $\mathcal{H}$, let P_n denote the orthogonal projection onto the subspace generated by $e_0, ..., e_n$, and let W_α be a hyponormal weighted shift with a weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. For $s \in \mathbf{C}$, we let $D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]$. For $n \geq 0$, let

$$\begin{aligned} D_n(s) &= P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n \\ &= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_k &: = u_k + |s|^2 v_k, \\ r_k &: = s\sqrt{w_k}, \\ u_k &: = \alpha_k^2 - \alpha_{k-1}^2, \\ v_k &: = \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2, \\ w_k &: = \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2 \quad (k \geq 0), \end{aligned}$$

and $\alpha_{-1} = \alpha_{-2} := 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbf{C}$ and every $n \geq 0$. To detect this positivity, we consider $d_n(\cdot) := \det(D_n(\cdot))$. By direct computation we have

$$\begin{aligned} d_0 &= q_0, \\ d_1 &= q_0 q_1 - |r_0|^2, \\ d_{n+2} &= q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n \qquad (n \geq 0); \end{aligned}$$

by inspection, d_n is actually a polynomial in $t := |s|^2$ of degree $n+1$, with McLaurin expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i) t^i.$$

To detect the positivity of $d_n(t)$, the following concept was introduced.

Definition 2.1 ([5])**.** Let $\alpha : \alpha_0, \alpha_1, \ldots$ be a weight sequence. We say that W_α is *positively quadratically hyponormal* if $c\,(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$. Recall that a weighted shift W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$.

To detect the structure of quadraticality, we consider the positively quadratical hyponormality.

Let $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3}, \cdots$ be a weight sequence with $\alpha_i \in \mathbb{R}_+ \setminus \{0\}$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. We first characterize quadratically hyponormal weighted shifts.

Theorem 2.2 ([13])**.** *Let W_α be a hyponormal weighted shift with a weight sequence α. Then the following statements are equivalent*:

(i) *W_α is quadratically hyponormal*;

(ii) *for any $s \in \mathbb{C}$, $x_i \in \mathbb{C}$, and $i = 1, 2, \cdots, n$ $(n \geq 2)$,*

$$\sum_{i=0}^{n} u_i |x_i|^2 + \sum_{i=0}^{n-1} \sqrt{w_i}(s x_i \bar{x}_{i+1} + \bar{s}\bar{x}_i x_{i+1}) + \sum_{i=0}^{n} v_i |s|^2 |x_i|^2 \geq 0;$$

(iii) *for any* $s \in \mathbb{R}_+$, $x_i \in \mathbb{R}_+$, *and* $i = 1, 2, \cdots, n$ $(n \geq 2)$,

$$\sum_{i=0}^{n} u_i x_i^2 - 2s \sum_{i=0}^{n-1} \sqrt{w_i} x_i x_{i+1} + s^2 \sum_{i=0}^{n} v_i x_i^2 \geq 0;$$

(iv) *for any* $x_i \in \mathbb{R}_+$, $s \in \mathbb{R}_+$ *and* $i = 1, 2, \cdots, n$ $(n \geq 2)$,

$$\sum_{i=0}^{n} q_i x_i^2 - 2 \sum_{i=0}^{n-1} r_i x_i x_{i+1} \geq 0$$

(*note that the real variable* s *appears in* q_i *and* r_i).

Let $0 < a \leq b \leq c$ and let $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $0 < x \leq a$. We write $W_{\alpha(x)}$ for the weighted shift with a weight sequence $\alpha(x)$ extended from a recursive weight sequence $\alpha : (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ throughout this article. Recall from [5, Th. 4.3] that :

$$h_2^+ := h_2^+(\alpha) = \min \left\{ \sqrt{a}, \left(\frac{a^2 b^2 c + ab^2(c-a)K + ab(c-b)K^2}{a^3 b + ab(c-a)K + (a^2 + bc - 2ab)K^2} \right)^{1/2} \right\},$$

where

$$\begin{aligned} K &: = -\frac{\psi_1^2 L^2}{\psi_0}, \\ L^2 &: = \frac{\psi_1 + \sqrt{\psi_1^2 + 4\psi_0}}{2} = \|W_{\alpha(x)}\|^2, \\ \psi_1 &: = \frac{b(c-a)}{b-a}, \ \psi_0 := -\frac{ab(c-b)}{b-a}. \end{aligned}$$

The following is a relationship between the positive quadraticality and the quadraticality in the case of recursively weighted shifts.

Theorem 2.3 ([13]). *Let* $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ *with* $0 < x \leq a \leq b \leq c$. *Then the weighted shift* $W_{\alpha(x)}$ *is quadratically hyponormal if and only if it is positively quadratically hyponormal.*

The following problem is motivated from Theorem 2.3.

Problem 2.4. Let $\alpha : \sqrt{x_1}, ..., \sqrt{x_n}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $0 < x_1 \leq \cdots \leq x_n \leq a \leq b \leq c$. Is it true that the weighted shift W_α is quadratically hyponormal if and only if it is positively quadratically hyponormal? First try to solve the problem in the case of $n = 2$.

Let W_α be a hyponormal weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ with $\alpha_0 = \alpha_1 = 1$. It is well-known that W_α is 2-hyponormal, then W_α is flat (i.e., $\alpha_0 = \alpha_1 = \cdots$) (cf. [Cu1]). Also W_α is polynomially hyponormal, then W_α is flat. On the other hand, the sequence $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, ...$ induces a quadratically hyponormal weighted shift, which gives the following problem.

Problem 2.5. Describe all quadratically hyponormal weighted shifts with first two equal weights.

As a detection of Problem 2.5, we first focus on recursively generated weighted shifts of the form W_α, where $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ and $1 \le x \le y$. It is interesting to consider in detail the region

$$\mathcal{R} := \{(x, y) | W_\alpha \text{ is quadratically hyponormal}\}.$$

By [5, Theorem 4.3] and its proof, we have that W_α is quadratically hyponormal if and only if

$$1 \le h_2^+ := \left(\frac{x^2 y + x^2 (y-1)K + x(y-x)K^2}{x + x(y-1)K + (1+xy-2x)K^2} \right)^{1/2}.$$

By direct computation, we have that

$$\mathcal{R} = \{(x, y) | x(xy-1) + x(x-1)(y-1)K - (x-1)^2 K^2 \ge 0\},$$

where

$$K = \frac{x(y-1)^2 \left(x(y-1) + \sqrt{x^2(y-1)^2 - 4x(x-1)(y-x)} \right)}{2(x-1)^2(y-x)}.$$

The region $\mathcal{R}$ can be described as the following two theorems.

Theorem 2.6 ([6]). *$\mathcal{R} = \{(x, y) \,|\, 1 < x < y$ and $f(x, y) \ge 0\}$, where $f(x, y) = \sum_{i=0}^{7} \phi_i y^i$ with*

$$\begin{aligned}
\phi_0 &:= x^2 - 4x^3 + 5x^4 - x^5, \\
\phi_1 &:= -2x + 7x^2 - 2x^3 - 11x^4 + x^5, \\
\phi_2 &:= 1 - 2x - 7x^2 + 4x^3 + 29x^4 - 5x^5 + x^6, \\
\phi_3 &:= -2x + 22x^2 - 33x^3 - 22x^4, \\
\phi_4 &:= -11x^2 + 24x^3 + 25x^4 - 3x^5, \\
\phi_5 &:= 3x^2 - 12x^3 - 15x^4 + 3x^5, \\
\phi_6 &:= 4x^3 + 4x^4 - x^5, \\
\phi_7 &:= -x^3.
\end{aligned}$$

Theorem 2.7 ([6]). *The planar set $\mathcal{R}$ is a closed convex set with nonempty interior.*

In particular, we can draw the boundary of $\mathcal{R}$ in detail (see [6, Fig. 1]).

Since $\mathcal{R} := \{(x, y) | W_\alpha$ is quadratically hyponormal$\}$ is a closed set, there exist maximum values x_M and y_M of x and y such that $\mathcal{R} \cap (\{x_M\} \times \mathbf{R})$ and $\mathcal{R} \cap (\mathbf{R} \times \{y_M\})$ are singletons. So the following problem can be suggested.

Problem 2.8. Find a concrete expression for x_M and y_M.

As a slight general problem, one can consider the case of $\alpha : 1, 1, (\sqrt{x}, \sqrt{y}, \sqrt{z})^\wedge$ with $1 \le x \le y \le z$. It is clear that

$$\mathcal{V} := \{(x, y, z) | W_\alpha \text{ is quadratically hyponormal}\}$$

is a solid in 3-dimensional space.

Problem 2.9. Describe $\mathcal{V}$ and its boundary.

Problem 2.10. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $1 \le x \le a \le b \le c$. Find $\mathcal{W} := \{x | W_\alpha$ is quadratically hyponormal$\}$.

3. Cubically hyponormal weighted shifts

Let W_α be a hyponormal weighted shift with a weight sequence $\alpha := \{\alpha_i\}_{i=0}^{\infty}$. Recall that a weighted shift W_α is *cubically hyponormal* if

$$D(a,b) := \left[\left(W_\alpha + aW_\alpha^2 + bW_\alpha^3\right)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3\right] \geq 0 \text{ for any } a,b \in \mathbf{C}.$$

Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. We may consider $\mathcal{H}$ as $\ell^2(\mathbf{Z}_+)$, where $\ell_2 := \ell^2(\mathbf{Z}_+)$ is the set of sequare summable sequences in $\mathbf{C}$. Let P_n be the orthogonal projection on $\vee_{i=0}^{n}\{e_i\}$. For $a,b \in \mathbf{C}$ we let

$$D_n(a,b) := P_n \left[\left(W_\alpha + aW_\alpha^2 + bW_\alpha^3\right)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3\right]\Big| P_n\ell^2(\mathbf{Z}_+).$$

Then it is obvious that W_α is cubically hyponormal if and only if the matrix $D_n(a,b) \geq 0$ for any $a,b \in \mathbf{C}$ and any $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers.

Proposition 3.1 ([14]). *Let W_α be a hyponormal weighted shift. Then the following statements are equivalent*

(i) *W_α is cubically hyponormal*;

(ii) *for any a,b in $\mathbf{C}$ and any $\mathbf{x} \in \ell_2$, we have*

$$F(\mathbf{x},a,b) = |b|^2 p_0 |x_0|^2 + \left(\begin{bmatrix} |a|^2 v_0 & a\bar{b}\sqrt{t_0} \\ \bar{a}b\sqrt{t_0} & |b|^2 p_1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}\right) + \sum_{i=0}^{\infty}\left(\Delta_i \begin{bmatrix} x_i \\ \bar{a}x_{i+1} \\ \bar{b}x_{i+2} \end{bmatrix}, \begin{bmatrix} x_i \\ \bar{a}x_{i+1} \\ \bar{b}x_{i+2} \end{bmatrix}\right) \geq 0,$$

where

$$\Delta_i := \begin{bmatrix} u_i & \sqrt{w_i} & \sqrt{s_i} \\ \sqrt{w_i} & v_{i+1} & \sqrt{t_{i+1}} \\ \sqrt{s_i} & \sqrt{t_{i+1}} & p_{i+2} \end{bmatrix} \qquad (i \in \mathbf{N} \cup \{0\}).$$

Recall that for $a,b \in \mathbf{C}$,

$$\begin{aligned} D_n(a,b) &= P_n[(W_\alpha + aW_\alpha^2 + bW_\alpha^3)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3]\big| P_n\ell^2(\mathbf{Z}_+) \\ &= \begin{bmatrix} q_0 & r_0 & z_0 & 0 & & & & \\ \bar{r}_0 & q_1 & r_1 & z_1 & 0 & & & \\ \bar{z}_0 & \bar{r}_1 & q_2 & r_2 & z_2 & 0 & & \\ 0 & \bar{z}_1 & \bar{r}_2 & q_3 & r_3 & z_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & z_{n-2} \\ & & & & \ddots & \ddots & \ddots & \ddots & r_{n-1} \\ & & & & & 0 & \bar{z}_{n-2} & \bar{r}_{n-1} & q_n \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_n \quad &: \quad = (\alpha_n^2 - \alpha_{n-1}^2) + (\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-2}^2\alpha_{n-1}^2)\,|a|^2 \\ &\quad\quad +(\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2)\,|b|^2\,, \\ r_n \quad &: \quad = \alpha_n(\alpha_{n+1}^2 - \alpha_{n-1}^2)\bar{a} + \alpha_n(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)a\bar{b}, \\ z_n \quad &: \quad = \alpha_n\alpha_{n+1}(\alpha_{n+2}^2 - \alpha_{n-1}^2)\bar{b}. \end{aligned}$$

To detect $D_n(a,b) \geq 0$ for any $a, b \in \mathbf{C}$ and any $n \in \mathbf{N}$, we consider $d_n(a,b) := \det D_n(a,b)$. Hence if W_α is cubically hyponormal, then $d_n(a,b) \geq 0$ for any $a, b \in \mathbf{R}$ and any $n \in \mathbf{N}$. Hence the determinant $d_n(a,b)$ will be helpful for studying cubically hyponormal weighted shifts. For example, we suggest the following problem.

Problem 3.2. Discuss the flatness of cubically hyponormal weighted shifts such as that of quadratically hyponormal weighted shifts.

We now give distinction examples for the classes of cubically hyponormal, 2-hyponormal and quadratically hyponormal operators.

Theorem 3.3 ([14]). *Let $\alpha(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ be a weight sequence with Bergman tail. Then there exists δ with $\frac{9}{16} < \delta < \frac{2}{3}$ such that*

$$\begin{cases} W_{\alpha(x)} \text{ is cubical but not 2-hyponormal,} & \text{if } \frac{9}{16} < x \leq \delta, \\ \qquad \text{and} & \\ W_{\alpha(x)} \text{ is quadratical but not cubically hyponormal,} & \text{if } \delta < x \leq \frac{2}{3}. \end{cases}$$

If we take $x = 0.564$ instead of $\frac{9}{16}(= 0.5623)$ in the proof of Theorem 3.3, by the similar method it follows easily that $G(a, b, 0.564) > 0$. Hence δ appearing in Theorem 3.3 should be bigger than 0.564. So we may state the following corollary as a concrete example.

Corollary 3.4. *Let $\alpha : \sqrt{\frac{141}{250}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$ be a weight sequence with Bergman tail. Then W_α is cubically hyponormal but not 2-hyponormal.*

Problem 3.5. Find the exact value δ in Theorem 3.3.

The following is an anologous to Theorem 2.6 in the case of cubically hyponormal weighted shifts.

Problem 3.6. Let $\alpha : 1, (1, \sqrt{a}, \sqrt{b})^\wedge$ with $0 < a < b$. Find the set $\mathcal{W} := \{x | W_\alpha$ is cubically hyponormal$\}$.

It seems to me that the method of Theorem 3.2 can prove the following problem without difficulty. We remain it as a problem here.

Problem 3.7. Find a weighted shift W_α which is quartically hyponormal but not 2-hyponormal.

More generally, we introduce the following hard problem.

Problem 3.8 (cf. [9], [10]). Find a weighted shift W_α which is polynomially hyponormal but not 2-hyponormal.

In fact, the following fundamental problem have not known yet. Some ideas can be found in [12], [13] and [14].

Problem 3.9. For given any $k \in \mathbf{N}$, find a weighted shift W_α which is weakly k-hyponormal but not k-hyponormal.

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