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A CONVEXITY THEOREM FOR THREE TANGLED HAMILTONIAN TORUS ACTIONS

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Abstract. The convexity theorem for hamiltonian torus actions states that the image of a moment map of a hamiltonian torus action on a compact, connected symplectic manifold is a convex polytope ([A], [G-S]). Kirwan generalized this theorem to the case of any compact, connected Lie group, which also gives us a convex polytope ([K]). On the other hand, if a torus which has half the dimension of the manifold acts effectively in a hamiltonian fashion, then its moment map provides a completely integrable system. Many important completely integrable systems are super-integrable systems in which each trajectory is linear in a smaller torus than the Liouville torus: the harmonic oscillators, the Kepler system, the Toda lattice, etc. In this note, motivated by the structure of the super-integrable system on the Toda lattice ([A-D-S]), we shall give a new generalization of the convexity theorem for hamiltonian torus actions.

1. Backgrounds and Results

In this note, we will give a new generalization of the convexity theorem in symplectic geometry as an approach to a special integrable systems which are called super-integrable systems. We explain some backgrounds and our results first, and then provide an example which illustrates our result.

A *symplectic manifold* (M, ω) is a pair of a smooth manifold M and a nondegenerate differential 2-form ω on M. By Darboux's theorem, ω locally looks like the standard symplectic form on even dimensional Euclidian space. This makes the relations with the classical mechanics in physics, and shows that peculiar properties for a symplectic manifold are global (or topological) ones. To analyze a physical system, it is very important to find its symmetry since symmetry give us some conserved quantities of equations of motions. It is also important and interesting to play with those symmetries on symplectic manifolds, from the point of view of geometry.

Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. *A hamiltonian G-action on* (M, ω) is a G-action on M which preserves the symplectic form ω, and has a G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that

$$\omega(X^\sharp, \cdot) = d\mu^X \quad (X \in \mathfrak{g}),$$

where $X^\sharp$ is the vector field on M generated by the infinitesimal action by X, and μ^X is a function on M defined by the point wise pairing of μ and X. μ is called a *moment map.* A moment map of a hamiltonian action on a symplectic manifold can be understood as a map composed of conserved quantities on the manifold in the sense of classical mechanics. If the manifold is compact and the group is a torus, we can regard a moment map as a completely integrable system.

Atiyah and Guillemin-Sternberg proved that the image of a moment map of a hamiltonian torus action on a compact, connected symplectic manifold is a convex polytope ([A], [G-S]). This theorem opened up some connections between symplectic geometry and combinatorics.

Theorem 1.1 (Atiyah [A], Guillemin-Sternberg [G-S])**.** *Let (M,ω) be a compact, connected symplectic manifold. Let T be a torus, and let $\mathfrak{t}$ be its Lie algebra. Suppose that $\psi : T \to \mathrm{Sympl}(M,\omega)$ is a hamiltonian action with a moment map $\mu : M \to \mathfrak{t}^*$. Then the image of μ is the convex hull of the image of the fixed point set of the action.*

Kirwan gave a beautiful generalization of Theorem 1.1 for hamiltonian group actions for compact, connected Lie groups ([K]).

Theorem 1.2 (Kirwan [K])**.** *Let (M,ω) be a compact, connected symplectic manifold. Let G be a compact, connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. Suppose that $\psi : G \to \mathrm{Sympl}(M,\omega)$ is a hamiltonian action with a moment map $\mu : M \to \mathfrak{g}^*$. Then the intersection of the image of μ and the positive Wyel chamber is a convex polytope.*

There is a concept of integrable systems in a non-commutative sense, which is called *super-integrable systems.* Roughly speaking, a super-integrable system is an integrable system in which each trajectory is linear in a smaller torus than the Liouville tori. In terms of conserved quantities, we can think of a super-integrable system as an integrable system with $\frac{1}{2}\mathrm{dim}M$ commutative conserved quantities which also has k $(1 \leq k \leq \frac{1}{2}\mathrm{dim}M - 1)$ extra conserved quantities where the total $\frac{1}{2}\mathrm{dim}M + k$ functions are independent (one can assign additional conditions on them for some purposes).

Many important completely integrable systems are super-integrable systems: harmonic oscillators, the Kepler sysetem, the Euler top, the Toda lattice, etc. In the case of $k = \frac{1}{2}\mathrm{dim}M - 1$ (which is called maximally super-integrable), there are $\mathrm{dim}M - 1$ conserved quantities, and this shows that each trajectory has to be periodic.

In a super integrable system, the target space of the map composed of $\frac{1}{2}\mathrm{dim}M+k$ conserved quantities may not form a Lie subalgebra of the symmetry of the system. However, in the maximally super-integrable system in Toda lattice which is shown in [A-D-S], the extra conserved quantities are also commutative, and share one function with the original integrable system. That is to say, the target space can be interpreted as a sum of two commutative subalgebra of the Lie algebra of the symmetry with one dimensional intersection. What can be known for the image of the map defined by these $\mathrm{dim}M-1$ conserved quantities? In the following theorem, we give a different kind of generalization of Theorem 1.1 compared to Theorem 1.2, as an approach for this question.

Theorem 1.3. *Let (M,ω) be a compact, connected symplectic manifold, let G be a compact Lie group, and let $\psi : G \to \mathrm{Sympl}(M,\omega)$ be a hamiltonian action with a* *moment a map $\mu : M \to \mathfrak{g}^*$. Assume that the Lie algebras $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$ of maximal tori T_1, T_2, T_3 of G satisfy the condition*

$$(*) \qquad \mathfrak{t}_i = \mathfrak{t}_1 \cap \mathfrak{t}_2 \cap \mathfrak{t}_3 + [\mathfrak{t}_j, \mathfrak{t}_k] \qquad (\{i, j, k\} = \{1, 2, 3\}).$$

Let $\mathcal{R}_{ij} : \mathfrak{g}^* \to (\mathfrak{t}_i + \mathfrak{t}_j)^*$ be the natural restriction map. Then for any $\{i, j, k\} = \{1, 2, 3\}$, the image of $\mathcal{R}_{ij} \circ \mu : M \to (\mathfrak{t}_i + \mathfrak{t}_j)^*$ is the convex hull of the $\mathrm{Ad}^*(T_k)$-orbit of the image of the fixed point set of the T_i-action.

Remark. We recover Theorem 1.1 with a maximal torus $T_1 = T_2 = T_3$.

Theorem 1.1 also states that the image of the moment map can be described by the fixed point set of the torus action. Theorem 1.3 is a generalization in this sense, too.

A proof of this theorem will be published elsewhere in future, and our proof will use the statement of Theorem 1.1 itself to prove Theorem 1.3.

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2. An Example

2.1. **Example: $\mathbb{C}P^2$ as a hamiltonian $\mathbb{T}\times SU(2)$-space.** The complex projective plane $\mathbb{C}P^2$ is a symplectic manifold as a coadjoint orbit of $SU(3)$. Let us define the subgroup $\mathbb{T} \times SU(2)$ of $SU(3)$ by

$$\mathbb{T} \times SU(2) = \left\{ \left(\begin{array}{c|cc} e^{-2i\zeta} & 0 & 0 \\ \hline 0 & e^{i\zeta} & 0 \\ 0 & 0 & e^{i\zeta} \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{array}\right) \in SU(3) \,\middle|\, \zeta \in \mathbb{R}, B \in SU(2) \right\}.$$

Then the corresponding Lie subalgebra is

$$\mathbb{R} \oplus \mathfrak{su}(2) = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{Y}} \\ 0 & & \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta \in \mathbb{R}, Y \in \mathfrak{su}(2) \right\}.$$

Let us define a $SU(3)$-invariant inner product $\phi : \mathfrak{su}(3) \times \mathfrak{su}(3) \to \mathbb{R}$ by

$$\phi(X, Y) = \mathrm{Re}\mathrm{Tr}(X^* Y) \quad (X, Y \in \mathfrak{su}(3)),$$

then the restriction of this inner product on $\mathbb{R} \oplus \mathfrak{su}(2)$ defines a $\mathbb{T} \times SU(2)$-invariant inner product. By this inner product, we can identify the Lie algebra $\mathbb{R} \oplus \mathfrak{su}(2)$ and its dual. Under this identification, the map $\widetilde{\mu} : \mathbb{C}P^2 \longrightarrow \mathbb{R} \oplus \mathfrak{su}(2)$ defined by

$$\widetilde{\mu}([z]) = -\frac{i}{2|z|^2} \left(\begin{array}{c|cc} |z_0|^2 - |z|^2/3 & 0 & 0 \\ \hline 0 & |z_1|^2 - |z|^2/3 & z_1\bar{z_2} \\ 0 & z_2\bar{z_1} & |z_2|^2 - |z|^2/3 \end{array}\right) \quad (z \in \mathbb{C}^3 - \{\boldsymbol{o}\})$$

is a moment map of the $\mathbb{T} \times SU(2)$-action on $\mathbb{C}P^2$. Let us define

$$\mathfrak{t}_1 = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & i\theta & 0 \\ 0 & 0 & -i\theta \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \theta \in \mathbb{R} \right\},$$

$$\mathfrak{t}_2 = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -\eta \\ 0 & \eta & 0 \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \eta \in \mathbb{R} \right\},$$

$$\mathfrak{t}_3 = \left\{ \left(\begin{array}{c|cc} -2i\zeta & 0 & 0 \\ \hline 0 & i\zeta & 0 \\ 0 & 0 & i\zeta \end{array}\right) + \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & i\xi \\ 0 & i\xi & 0 \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \zeta, \xi \in \mathbb{R} \right\},$$

and let

$$T_1 = \exp \mathfrak{t}_1, T_2 = \exp \mathfrak{t}_2, T_3 = \exp \mathfrak{t}_3.$$

Then T_1, T_2, and T_3 are maximal tori of $\mathbb{T} \times SU(2)$, and it is easy to show that these $\mathfrak{t}_1, \mathfrak{t}_2$ and $\mathfrak{t}_3$ satisfy the assumption of Theorem 1.3. Under the identification of the Lie algebra $\mathbb{R} \oplus \mathfrak{su}(2)$ and its dual, the natural restriction map $\mathcal{R}_{12} : \mathfrak{g}^* \to (\mathfrak{t}_1 + \mathfrak{t}_2)^*$ corresponds to the orthogonal projection $\pi_{12} : \mathbb{R} \oplus \mathfrak{su}(2) \to \mathfrak{t}_1 + \mathfrak{t}_2$. Now, the map $\pi_{12} \circ \widetilde{\mu} : \mathbb{C}P^2 \longrightarrow \mathfrak{t}_1 + \mathfrak{t}_2$ is given by

$$\pi_{12} \circ \widetilde{\mu}([z]) = -\frac{i}{2|z|^2} \left(\begin{array}{c|cc} |z_0|^2 - |z|^2/3 & 0 & 0 \\ \hline 0 & |z_1|^2 - |z|^2/3 & i\mathrm{Im}(z_1\bar{z}_2) \\ 0 & -i\mathrm{Im}(z_1\bar{z}_2) & |z_2|^2 - |z|^2/3 \end{array}\right) \quad (z \in \mathbb{C}^3 - \{\boldsymbol{o}\}).$$

Let us calculate the image $\pi_{12} \circ \widetilde{\mu}(\mathbb{C}P^2)$. The fixed point set of T_1-action is

$$\mathrm{Fix}T_1 = \{[1,0,0], [0,1,0], [0,0,1]\}.$$

By direct calculation, we obtain

$$\begin{aligned} &\mathrm{Ad}(T)(\pi_{12} \circ \widetilde{\mu}(\mathrm{Fix}T_1)) \\ &\quad = \left\{ \frac{1}{6} \left(\begin{array}{c|cc} -2i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & i \end{array}\right) \right\} \\ &\qquad \cup \left\{ -\frac{1}{12} \left(\begin{array}{c|cc} -2i & 0 & 0 \\ \hline 0 & i & 0 \\ 0 & 0 & i \end{array}\right) - \frac{1}{4} \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & i\cos 2\xi & \sin 2\xi \\ 0 & -\sin 2\xi & -i\cos 2\xi \end{array}\right) \in \mathfrak{su}(3) \,\middle|\, \xi \in \mathbb{R} \right\}. \end{aligned}$$

Since this set is the union of a point and a circle in the 3-dimensional space $\mathfrak{t}_1 + \mathfrak{t}_2$, by Theorem 1.3, the image $\pi_{12} \circ \mu(\mathbb{C}P^2)$ is a cone with its interior and boundary which is given by the convex hull of the point and the circle (Figure 1).

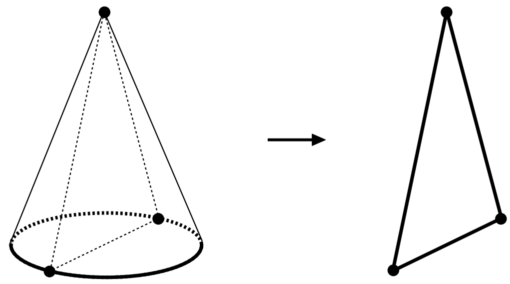

FIGURE 1. $\pi_{12} \circ \widetilde{\mu}(\mathbb{C}P^2)$ and the moment polytope of T_1

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