Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 2, December 2001, Pages 33–38

FOR THE BEHAVIOR OF THE SOLUTIONS OF A GENERAL POROUS MEDIA EQUATIONS AT LARGE TIME SCALE

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Abstract. We consider a general porous media equation $\varepsilon u_{tt} + u_t = \Delta u^m$,which is obtained by using a general form of Darcy's law , $\varepsilon q_t + \vec{q} = -k\nabla p$, where ε is a positive constant and show in one dimensional space that its solutions behaviors like solutions of a porous media equation , $u_t = \Delta u^m$, at large time scale.

1. Introduction

The theory of fluid flow in porous media is based upon the empirically determined Darcy's law while on the other the theories of fluid flow in classical hydrodynamics are based upon equations of motion.Such a Darcy's law

$$\vec{q} = -\frac{\kappa}{\mu}\left(\vec{\nabla}p + \rho\vec{g}\right) \tag{1}$$

where p is the pressure, ρ the density of the fluid (or gas), $\vec{g} = (0, 0, g)$ the vector giving the acceleration due to gravity, here assumed to be constant and κ is the permeability tensor which in most applications is assumed to be strictly positive definite. $\vec{q}$ is the volumetric flow rate , also called the seepage velocity and $\mu > 0$ is the dynamic viscosity of the fluid. Equation (1) is combined with the equation of continuity, or conservation of mass,

$$\phi\rho_t = -\mathrm{div}(\rho\vec{q}), \tag{2}$$

where ϕ is the porosity of the medium and an equation of state, or constitutive relation, for the gas which I take to be

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{\lambda}} \tag{3}$$

here $p_0, \rho_0 \in R^+$, and $\lambda \in [1, \infty)$ are The constant $\lambda > 1$ occurs when one assumes the expansion of the gas to be adiabatic. Then we obtain a porous media equation

$$\rho_t = \Delta\rho^m \quad (m > 1). \tag{4}$$

This equation is known as the porous media equation which models the density distribution of a gas in a porous medium. The equation is parabolic at any point (x, t) at which $\rho > 0$. However, at a point where $\rho = 0$, it is degenerate parabolic.Its initial-boundary problem fails to be classical solutions at the degenerated points. Thus, we need a concept of weak solutions. Classes of weak solutions for the

2000 *Mathematics Subject Classification.* 35K.
Key words and phrases. degenerate parabolic, General porous media equation.
Received May 8, 2001.

problems were introduced by Oleinik [5]. There they proved the existence and uniqueness of such solutions and showed that if at some constant t_0 a weak solution $\rho(x, t_0)$ has compact support, then $\rho(x,t)$ has compact support for any $t \geq t_0$. Moreover, they showed that in a neighborhood of any point (x,t) where $\rho > 0, \rho$ is a classical solution of the problem. Some explicit solutions of the porous media equation were found and these were all self similar solutions. Here the porous media equation is induced by Darcy's equation (1) under conditions of steady flow of incompressible fluids in one direction with neglecting gravity.However by using equation of continuity and motion, we find a general Darcy's equation

$$\varepsilon q_t + \vec{q} = -k\nabla p \tag{5}$$

where ε is a positive constant, which is applied for fluid flow in unsteady flow in porous medium though some restrictions are given. Thus we obtain $\varepsilon\rho_{tt} + \rho_t = (\rho^m)_\lambda$. The equation is also parabolic at any point (x,t) at which $\rho > 0$. However, at a point where $\rho = 0$, it is also degenerate parabolic. Thus we try to find its explicit self-similar solutions and will find that the solutions behavoir like the similarity solutions of (4) at large time scales.

2. Derivation of a general Darcy's equation and a general porous media equation

q is the volume flux and **v** the velocity for fluid flow in a porous medium with volume **V**. The two are related by the equality

$$\mathbf{q} = \phi\mathbf{v}. \tag{6}$$

where ϕ denotes the porosity of the porous medium. In the classical theory of fluid mechanics $\mathbf{q} \equiv \mathbf{v}$ since $\phi \equiv 1$. In dealing with unsaturated flow in a porous medium, a function $\mathbf{S}$, the saturation, is defined as a set function in the same way as ϕ.(see Fuiks[1]). The relationship between velocity and volume flux becomes $\mathbf{q} = (\phi\mathbf{S})\mathbf{v}$. We simply take $\mathbf{S} \equiv 1$. Let us first recall (see Serrin [3]) for a volume $\mathbf{V}(t)$, which depend on t,

$$\frac{d}{dt}\int_{\mathbf{V}(t)} f(x,t)\,dx = \int_{\mathbf{V}(t)} \frac{\partial}{\partial t} f(x,t)\,dx + \int_{\partial\mathbf{V}(t)} f(x,t)\mathbf{v}\cdot\nu dS_t, \tag{7}$$

where ν is the exterior unit normal to $\partial\mathbf{V}(t)$ and $\mathbf{v}$ is the velocity of the front. Then by balance of the momentum , the equation of motion as following is obtained.

$$\rho\phi\frac{\partial\mathbf{v}}{\partial t} + \rho\phi\mathbf{v}\nabla\mathbf{v} = \mathbf{f} + \sigma(\mathbf{q}) + \mathrm{div}\mathbf{T}. \tag{8}$$

where we denote the exterior body forces per unit volume by $\mathbf{f}$ and the body force per unit volume arising from the resistance of the medium to the flow by $\sigma = \sigma(\mathbf{q})$ and $T = (T_{ij})$ is a tensor representing the forces acting on the surface of the element. Here let us assume that the only exterior body force is the force of gravity. If we let $-\mathbf{g}$ denote the acceleration due to gravity, then the exterior body force due to gravity in a fluid element $V(t)$ is $-\int_{V(t)} \rho\phi\mathbf{g}\,dx$, so that the force per unit volume is $-\rho\phi\mathbf{g}$. Next ,let us assume that the surface forces act normal to the surface of the volume element,$V(t)$. The surface forces are then caused by the pressure exerted on the fluid particles on $\partial V(t)$ by adjacent fluid elements. Since the area of fluid on $\partial V(t)$ which can be acted upon is , infinitesimally , ρdS_t , we may take $T = -p\phi E$, where E is the identity tensor. Finally let us assume σ depends linearly on $\mathbf{q}$, i.e. there exits a positive definite matrix $\mathbf{K}$ such that

$$\rho\phi\frac{\partial \mathbf{v}}{\partial t} + \rho\phi\mathbf{v}\nabla\mathbf{v} = -\rho\phi\mathbf{g} - \nabla(p\phi) - \mathbf{K}\mathbf{q}. \tag{9}$$

Under conditions like steady flow in one direction, then $\frac{\partial \mathbf{v}}{\partial t} = 0, \mathrm{div}\mathbf{q} = 0$. Thus Darcy's law (1) is obtained. However if we consider it under situations in unsteady flow but divergence at the surface of the given volume is given as zero value, which can be considered in case of unsteady flow in a piped porous medium, then $\mathbf{v}\nabla\mathbf{v} = 0$. Thus from (9) , we have a general Darcy's equation

$$\varepsilon\mathbf{q}_t + \vec{q} = -k\nabla\vec{p} \tag{10}$$

where $k = \phi\mathbf{K}^{-1}$ and $\varepsilon = \rho\mathbf{K}^{-1}$. Here we assume that $\rho/\rho_0 = 1+\theta$ where θ is small. Then $\rho = \rho_0(1+\theta)$. Thus, $\mathrm{div}\,(\rho q) = \rho_0 \mathrm{div}\, q + \rho_0 \mathrm{div}\,\theta q$.
Here we neglect the term $\rho_0\mathrm{div}\,\theta q$ so $\mathrm{div}\,(\rho q) = \rho_0\mathrm{div}\, q$. By conservation of mass, (2), we have

$$\phi\rho_t = -\rho_0 \mathrm{div}\, q. \tag{11}$$

By differentiating both sides of (11) with respect to t we have

$$\phi\rho_{tt} = -\rho_0 \mathrm{div}\, q_t. \tag{12}$$

By taking the divergence in (5), we have

$$\varepsilon \mathrm{div} q_t + \mathrm{div} q = -k\Delta p. \tag{13}$$

Therefore, by inserting (12) and (11) into (13), we obtain

$$-\tfrac{\phi\varepsilon}{\rho_0}\rho_{tt} - \tfrac{\phi}{\rho_0}\rho_t = -k\Delta p. \tag{14}$$

From the equation of state, (3), $p = p_0(\rho/\rho_0)^{1/\lambda}$. By inserting this into(14),

$$\tfrac{\phi\varepsilon}{\rho_0}\rho_{tt} + \tfrac{\phi}{\rho_0}\rho_t = \tfrac{k\,p_0}{(\rho_0)^{1/2}}\Delta\rho^{1/\lambda}. \tag{15}$$

Hence since $\rho \approx \rho_0$,

$$\varepsilon\rho_{tt} + \rho_t = \tfrac{k\,p_0\rho_0}{\phi(\rho_0)^{1/\lambda}}\Delta\rho^{\frac{1}{\lambda}} \approx \tfrac{k\,p_0}{\phi\rho_0^{1/\lambda}(\lambda+1)}\Delta\rho^{(\lambda+1)/\lambda} \tag{16}$$

By rescaling we can take $k\,p_0/\phi\rho_0^{1/\lambda}(\lambda+1)$ to be equal to 1 and we let $(\lambda+1)/\lambda = m$. Then we obtain

$$\varepsilon\rho_{tt} + \rho_t = (\rho^m)_{xx}.$$

Replace ρ by u so that the standard form of the equation is

$$\varepsilon u_{tt} + u_t = (u^m)_{xx}. \tag{17}$$

3. A similarity solution of a general porous media equation at large time scale

In order to find a solution to (17), let us try a similar solution of the following type :

$$u(x,t) = (t+\tau)^\beta f(\zeta), \qquad \zeta = x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}}$$

where $\tau \in R$ is arbitrarily , $t+\tau > 0$. Then by differentiating the $u(x,t)$ term with respect to t, we have

$$u_t = \beta(t+\tau)^{\beta-1}f(\zeta) + (t+\tau)^\beta f'(\zeta)\tfrac{\partial\zeta}{\partial t}. \tag{18}$$

and

$$u_{tt} = \beta(\beta-1)(t+\tau)^{\beta-2}f(\zeta) + \beta(t+\tau)^{\beta-1}\frac{\partial\zeta}{\partial t}f'(\zeta) + \left\{\beta(t+\tau)^{\beta-1}\frac{\partial\zeta}{\partial t} + (t+\tau)^{\beta}\frac{\partial^2\zeta}{\partial t}\right\}f'(\zeta) + (t+\tau)^{\beta}\left(\frac{\partial\zeta}{\partial t}\right)^2 f''(\zeta) \tag{19}$$

and

$$(u^m)_{xx} = (f^m)''(\zeta)(t+\tau)^{\beta-1}. \tag{20}$$

By inserting (18),(19) , and (20) into (17), and dividing by $(t+\tau)^{\beta-1}$, we have

$$(f^m)'' - \varepsilon(t+\tau)\left(\frac{\partial\zeta}{\partial t}\right)^2 f'' - \left\{2\varepsilon\beta\frac{\partial\zeta}{\partial t} + \varepsilon(t+\tau)\frac{\partial^2\zeta}{\partial t} + (t+\tau)\frac{\partial\zeta}{\partial t}\right\}f' - \{\varepsilon\beta(\beta-1)(t+\tau)^{-1} + \beta\}f(\zeta) = 0 \tag{21}$$

Also, from $\zeta = x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}}$, we obtain

$$\begin{aligned} \frac{\partial\zeta}{\partial t} &= \frac{1}{2}\{1+(m-1)\beta\}\,x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}-1} \\ \frac{\partial^2\zeta}{\partial t^2} &= \{\frac{1}{4}\{1+(m-1)\beta\}^2 \\ &\quad + \frac{1}{2}\{1+(m-1)\beta\}\}x(t+\tau)^{-\frac{1}{2}\{1+(m-1)\beta\}-2} \end{aligned} \tag{22}$$

Then by inserting (22) into (21), we have

$$\begin{aligned} &(f^m)'' - \frac{\varepsilon}{4}\{1+(m-1)\beta\}^2\zeta^2(t+\tau)^{-1}f'' \\ &- \zeta(-\varepsilon\beta\{1+(m-1)\beta\}(t+\tau)^{-1} - \frac{1}{2}\{1+(m-1)\beta\}.+ \\ &\varepsilon[\frac{1}{4}\{1+(m-1)\beta\}^2 + \frac{1}{2}\{1+(m-1)\beta\}](t+\tau)^{-1})f' \\ &- (\varepsilon\beta(\beta-1)(t+\tau)^{-1} + \beta)f = 0. \end{aligned} \tag{23}$$

Then the equation (23) can be rewritten as follows

$$(f^m)'' + \frac{1}{2}\{1+(m-1)\beta\}\psi\zeta f' - \beta f = \Lambda_1 f^{''} + \Lambda_2 f' + \Lambda_3 f \tag{24}$$

where $\Lambda_1 = \frac{\varepsilon}{4}\{1+(m-1)\beta\}^2\zeta^2(t+\tau)^{-1}$,

$$\Lambda_2 = -\zeta(-\varepsilon\beta\{1+(m-1)\beta\}(t+\tau)^{-1} + \varepsilon\left[\frac{1}{4}\{1+(m-1)\beta\}^2 + \frac{1}{2}\{1+(m-1)\beta\}\right](t+\tau)^{-1}) \tag{25}$$

and $\Lambda_3 = \varepsilon\beta(\beta-1)(t+\tau)^{-1}$ If we let T be sufficiently large so that for all $t \geq T$, $\frac{1}{t+\tau} \approx 0$. Then Λ_1, Λ_2 ,and Λ_3 can be considered as sufficiently small as be negligible. Thus at sufficiently large time scale (23) becomes

$$(f^m)'' + \tfrac{1}{2}\{1+(m-1)\beta\}\zeta f' = \beta f(\zeta). \tag{26}$$

If we impose the boundary condition $f(0) = K,\ f(\infty) = 0$, $K > 0$ constant on (26), the solution $u(x,t)$ satisfies the lateral boundary condition $u(0,t) = (t+\tau)^{\alpha}K$ and $u(x,t) \to 0$ as $x \to \infty$ for fixed $t \in [0,T]$.

Let $\tilde{p} = (1 + (m-1)\beta)/2,\ \tilde{q} = \beta$. Then the following is obtained:

$$(f^m)'' + \tilde{p}\zeta f' = \tilde{q} f \tag{27}$$

$$f(0) = K, \quad f(\infty) = 0 \tag{28}$$

Now it is necessary to consider weak solutions to the problem (27) because I am looking for a weak solution of $u(x,t) = (t+\tau)^\beta f(\zeta)$.
Here a function f will be said to be a weak solution of (27) if

(i) f is bounded, continuous and nonnegative on $[0, \infty)$.
(ii) $(f^m)(\zeta)$ has a continuous derivative with respect to ζ on $[0, \infty)$.
(iii) f satisfies the identity

$$\int_0^\infty \phi' \{(f^m)' + \tilde{p}\zeta f\}\, d\zeta + (\tilde{p} + \tilde{q}) \int_0^\infty \phi f\, d\zeta = 0 \text{ for all } \phi \in C_0^1([0,\infty)).$$

Then the following theorem is known in Gilding [7] :
Theorem 1) when $K > 0$, the equation (27) has a weak solution with compact support if and only if $\tilde{p} \geq 0$ and $2\tilde{p} + \tilde{q} > 0$. This solution is unique. And, if we let $f(\zeta)$ be a weak solution of problem (27) with compact support, then the solution is of the form: $f(\zeta) > 0$ on $[0, a)$, $f(\zeta) = 0$ on $[a, \infty)$ for some $a > 0$.

Let us take $\beta = 1/(m-1) > 0$ for $m > 1$. Then (27) becomes

$$(f^m)'' + \zeta f' = \tfrac{1}{m-1} f(\zeta). \tag{29}$$

then (29) has a weak solution with compact support by the Theorem 1
The function

$$f(\zeta) = \begin{cases} \left\{ \left[\frac{m-1}{m}\right] a(a-\zeta) \right\}^{\frac{1}{m-1}}, & 0 \leq \zeta \leq a \\ 0, & a < \zeta < \infty \end{cases} \tag{30}$$

satisfies (29) with boundary conditions $f(0) = K,\ f(\infty) = 0$. Therefore,

$$u(x,t) = \begin{cases} (t+\tau)^{\frac{1}{m-1}} f(\zeta), & 0 \leq \zeta \leq a \\ 0, & a < \zeta < \infty \end{cases} \tag{31}$$

so

$$u(x,t) = \begin{cases} \left\{ \left[\frac{m-1}{m}\right] a[a(t+\tau) - x] \right\}^{\frac{1}{m-1}}, & 0 \leq x \leq a(t+\tau) \\ 0, & a(t+\tau) < x < \infty \end{cases} \tag{32}$$

This is the wave solution found in Oleinik [5] and is one of the similarity solutions of the equation $u_t = (u^m)_{xx}$ (see Gilding [7]). Thus for sufficiently large time scales, a similarity solution of $\varepsilon u_{tt} + u_t = (u^m)_{xx}$ reduces to the similarity solution of $u_t = (u^m)_{xx}$.

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