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A NEUMANN-DIRICHLET PRECONDITIONER FOR A FETI-DP FORMULATION WITH MORTAR METHODS

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ABSTRACT. In this article, we review a dual-primal FETI (FETI-DP) method with mortar methods. The mortar matching condition is used as the continuity constraints for the FETI-DP formulation. A Neumann-Dirichlet preconditioner is investigated and it is shown that the condition number of the preconditioned FETI-DP operator for the two-dimensional elliptic problem is bounded by $C\max_{i=1,\dots,N}\{(1+\log(H_i/h_i))^2\}$, where H_i and h_i are sizes of subdomain and mesh for each subdomain, respectively, and C is a constant independent of H_i and h_i. For the three-dimensional elliptic problem and the two-dimensional Stokes problem, edge average constraints are further introduced as primal constraints to solve the problem correctly and to obtain a scalable FETI-DP algorithm. The Neumann-Dirichlet preconditioner is shown to give the same condition number bound as for the two-dimensional elliptic problem. For the three dimensional elasticity problem, a relatively large set of primal constraints, which include average and momentum constraints over interfaces (faces) as well as vertex constraints, is introduced. With the preconditioner, the same condition number bound is then obtained for the elasticity problems with discontinuous material parameters when only some faces are chosen as primal faces on which the average and momentum constraints is imposed.

1. Introduction

This article is concerned with a preconditioner for an iterative method for the parallel solution of the elliptic problem, the two-dimensional Stokes problem and the three-dimensional compressible elasticity problem with nonconforming discretizations. Of the many methods for nonmatching meshes, including [9] and [30], we consider the mortar method [4, 7, 33, 34].

Recently the dual-primal FETI (FETI-DP) method introduced by Farhat, Lesoinne, and Pierson [14] has been applied to mortar finite elements methods [11, 12, 31]. For FETI-DP methods on nonmatching grids, Dryja and Widlund [11] proposed a preconditioner, so-called Dirichlet preconditioner, which gives a condition number bound $C(1+\log(H/h))^2$ with the Neumann–Dirichlet ordering of substructures, where H and h denote the maximum diameter of subdomains and the minimum size of meshes of all subdomains, respectively. Moreover, in [12], they proposed

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a different preconditioner, which is similar to one in [21], and proved the condition number bound $C(1 + \log(H/h))^2$. However, the constant C in the condition number bound depends on the ratio of meshes between neighboring subdomains. This restriction is impractical when the coefficients of elliptic problems are highly discontinuous between subdomains (see Wohlmuth [34]).

In [18], a FETI-DP operator was formulated in a different way from that of Dryja and Widlund [11, 12] and a Neumann–Dirichlet preconditioner was proposed, which gives the condition number bound $C \max_{i=1,\cdots,N} \left\{(1 + \log(H_i/h_i))^2\right\}$ with the constant C not depending on the ratio of meshes between neighboring subdomains. The proposed preconditioner is similar to the previous FETI-DP preconditioners except that it solves local problems with Neumann boundary conditions on nonmortar interfaces and with a zero Dirichlet boundary condition on mortar interfaces. The additional complication caused by mortar discretizations can be handled by using this preconditioner.

The extension of the FETI-DP method in [18] has been done to the three dimensional problem [16]. In the FETI-DP formulation, we need redundant continuity constraints to get the same condition number bound as the two dimensional problem. The redundant constraints are that averages of the solution across subdomain interfaces are the same, which is so called face constraints in [24]. With the similar idea to the previous work in [18], a Neumann-Dirichlet preconditioner is proposed, and it is shown that the same condition number bound as the two-dimensional elliptic problem holds for the three-dimensional elliptic problems whose coefficients do not change rapidly across subdomain interfaces. Further, with an assumption on mesh sizes according to the magnitude of coefficients, we get the same condition number bound for elliptic problems with discontinuous constant coefficients. In this case, the constant C does not depend on the coefficients.

In [19] the FETI-DP algorithm developed in [18] was extended to the two-dimensional Stokes problem. The inf-sup stable $P_1(h) - P_0(2h)$ finite element space is considered in each subdomain. The mortar matching conditions are imposed on the velocity functions. An optimal approximation of mortar methods for the Stokes problem was proved by Belgacem [5]. If the inf-sup constant is independent of mesh sizes and subdomain sizes, then the optimal order of approximation follows independently of the number of subdomains and mesh sizes as in the case of elliptic problems. As in [16, 26], the primal constraints, i.e., edge average and vertex constraints, are introduced to solve the Stokes problem efficiently and correctly. Then a Neumann-Dirichlet preconditioner is proposed and the same condition number bound is analyzed.

The FETI-DP algorithm of [18] was extended to the three-dimensional compressible elasticity problem [17]. Klawonn and Widlund [22] considered various primal constraints for elasticity problems with discontinuous Lamé parameters. In their work, some faces and edges are selected as fully primal faces and fully primal edges. They work with edge average constraints on a fully primal face, and edge average and edge moment constraints on a fully primal edge. However, edge constraints are not compatible with mortar matching constraints. In [17], the face average and face moment constraints on the faces are introduced. Further, the number of primal constraints are reduced by selecting only some of the faces as primal faces for which the face average and face moment constraints are applied.

This article is organized as follows. In section 2, we derive a FETI-DP operator using the mortar matching condition as continuity constraints and propose a preconditioner for the two-dimensional elliptic problem. Furthermore we provide the condition number bound of the preconditioned FETI-DP operator. We do the same process for the three dimensional elliptic problem, the two-dimensional Stokes problem and the three-dimensional compressible elasticity problem in sections 3, 4 and 5, respectively.

Throughout this article, C denotes a generic constant independent of mesh sizes and subdomain sizes. We will use H_i and h_i to denote the subdomain size and the typical mesh size of each subdomain Ω_i, respectively.

2. Two-dimensional elliptic problem

2.1. **A model problem and Sobolev spaces.** Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and we consider a FETI-DP method on nonmatching grids for the following elliptic problem: For $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (A(x)\nabla u(x)) + \beta(x)u(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot (A(x)\nabla u(x)) &= 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{2.1}$$

Here, $A(x) = (\alpha_{ij}(x))$ for $i, j = 1, 2$ and $\boldsymbol{n}$ is the outward unit vector normal to Γ_N. We assume that $\alpha_{ij}(x), \beta(x) \in L^\infty(\Omega)$, $A(x)$ is uniformly elliptic, $\beta(x) \geq 0$ for all $x \in \Omega$, and $|\Gamma_D| \neq 0$, where $|\Gamma_D|$ denotes the measure of Γ_D.

Let Ω be partitioned into nonoverlapping polygonal subdomains $\{\Omega_i\}_{i=1}^N$. We assume that the partition is geometrically conforming, which means that the subdomains intersect with neighboring subdomains on a whole edge or at a vertex. Ω_i^h denotes a quasi-uniform triangulation of the subdomain Ω_i. Let d_τ be the diameter of $\tau \in \Omega_i^h$, and $h_i = \max_{\tau \in \Omega_i^h} d_\tau$. We note that the meshes need not match across the subdomain interfaces.

For each subdomain Ω_i, we introduce a finite element space

$$X_i := \{v \in H_D^1(\Omega_i) \ : \ v|_\tau \in P_1(\tau), \ \tau \in \Omega_i^h\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) \ : \ v = 0 \text{ on } \Gamma_D \cap \partial\Omega_i\}$ and $P_1(\tau)$ is a set of polynomials of degree ≤ 1 in τ. For $(u_i, v_i) \in X_i \times X_i$, define a bilinear form

$$a_i(u_i, v_i) := \int_{\Omega_i} A(x)\nabla u_i \cdot \nabla v_i \, dx + \int_{\Omega_i} \beta(x) u_i v_i \, dx.$$

To get a FETI-DP formulation, we need a finite element space in Ω:

$$X := \left\{ v \in \prod_{i=1}^N X_i \ : \ v \text{ is continuous at subdomain vertices} \right\}. \tag{2.2}$$

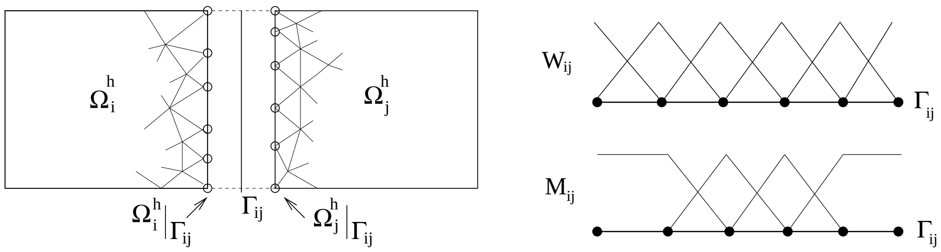

FIGURE 1. (a) Mortar and nonmortar sides of Γ_{ij}. (b) Basis functions for W_{ij} and M_{ij}.

By restricting the space X_i on the boundary of the subdomain Ω_i, we define

$$W_i := X_i|_{\partial\Omega_i} \ \forall i = 1, \ldots, N. \tag{2.3}$$

Then we let

$$W := \left\{ w \in \prod_{i=1}^{N} W_i \ : \ w \text{ is continuous at subdomain vertices} \right\}. \tag{2.4}$$

In this article, we will use the same notation for finite element functions and the corresponding vectors of nodal values. For example, w_i is used to denote a finite element function or the vector of nodal values of that function. The same applies to the notation for function spaces such as W_i, X, W, etc.

We define S^i as the Schur complement matrix obtained from the bilinear form $a_i(\cdot, \cdot)$ over the finite elements X_i (see page 50 in [29]). Using this operator, a seminorm is defined for $w_i \in W_i$:

$$|w_i|_{S^i}^2 := \langle S^i w_i, w_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the l^2-inner product of vectors. For $w \in W$, since w is continuous at subdomain vertices, by summing up these seminorms, we define a norm

$$\|w\|_W^2 := \sum_{i=1}^{N} |w_i|_{S^i}^2, \quad w_i = w|_{\partial\Omega_i}.$$

Moreover, we define a subspace of W

$$W_r := \{ w \in W \ : \ w \text{ vanishes at subdomain vertices} \}. \tag{2.5}$$

2.2. **Mortar matching conditions.** We note that the space X is not contained in $H^1(\Omega)$. To approximate the solution of the problem (2.1) in X, we use the mortar matching condition.

First, let $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$. For Γ_{ij} such that $|\Gamma_{ij}| \neq 0$, we distinguish $\Omega_i^h|_{\Gamma_{ij}}$ and $\Omega_j^h|_{\Gamma_{ij}}$, as in Figure 1(a). We assume that both sides have more than three nodal points including end points. Then we choose one as a nonmortar side and the other as a mortar side and define

$$m_i := \{ j \ : \ |\Gamma_{ij}| \neq 0, \ \Omega_j^h|_{\Gamma_{ij}} \text{ is a mortar side of } \Gamma_{ij} \},$$
$$s_i := \{ j \ : \ |\Gamma_{ij}| \neq 0, \ \Omega_j^h|_{\Gamma_{ij}} \text{ is a nonmortar side of } \Gamma_{ij} \}.$$

For $j \in m_i$, $\Omega_i^h|_{\Gamma_{ij}}$ is the nonmortar side of Γ_{ij} and from the finite elements on the nonmortar side, we get

$$W_{ij} := \{v|_{\Gamma_{ij}} \ : \ v \in X_i\} \ \ \forall\, j \in m_i.$$

Next, let

$$\{\phi_0^{ij},\ \phi_1^{ij}, \ldots, \phi_{N_{ij}}^{ij},\ \phi_{N_{ij}+1}^{ij}\}$$

be nodal basis functions for W_{ij}. Moreover, we assume that the basis functions are sequentially ordered according to the location of nodes on Γ_{ij}. Let (see Figure 1(b))

$$M_{ij} := \operatorname{span}\{\phi_0^{ij} + \phi_1^{ij},\ \phi_2^{ij}, \ldots, \phi_{N_{ij}-1}^{ij}, \phi_{N_{ij}}^{ij} + \phi_{N_{ij}+1}^{ij}\}.$$

Then we take the Lagrange multiplier space

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}. \tag{2.6}$$

Bernardi, Maday, and Patera [7] first introduced this type of Lagrange multiplier space. They imposed the following mortar matching condition on X, i.e., $v \in X$ satisfies

$$\int_{\Gamma_{ij}} (v_i - v_j)\lambda_{ij}\, ds = 0 \ \ \forall\, \lambda_{ij} \in M_{ij},\ i = 1, \ldots, N,\ j \in m_i. \tag{2.7}$$

For $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$, we denote $\partial\Omega_i \cap \partial\Omega_j$ as Γ_{ij} if $\Omega_i^h|_{\Gamma_{ij}}$ is a nonmortar side and as Γ_{ji}, otherwise. We assume that $\Omega_i^h|_{\Gamma_{ij}}$ is the nonmortar side and $\Omega_j^h|_{\Gamma_{ij}}$ is the mortar side of Γ_{ij}. Denote the basis for M_{ij} by $\{\xi_k^{ij}\}_{k=1}^{N_{ij}}$ and let $\{\phi_k^{ji}\}_{k=0}^{N_{ji}+1}$ be the basis functions for $W_j|_{\Gamma_{ij}}$. Define matrices B_i^{ij} and B_j^{ij} with entries

$$\left(B_i^{ij}\right)_{lk} = \int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ij}\, ds, \quad \forall l = 1, \ldots, N_{ij}, \forall k = 0, \ldots, N_{ij}+1,$$
$$\left(B_j^{ij}\right)_{lk} = -\int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ji}\, ds, \quad \forall l = 1, \ldots, N_{ij}, \forall k = 0, \ldots, N_{ji}+1.$$

Now define $E_{ij} : M_{ij} \to M$, an extension operator from M_{ij} to M by zero, and $R_{ij}^l : W_l \to W_l|_{\Gamma_{ij}}$ for $l = i, j$, a restriction operator. Let

$$\begin{aligned} B_i &= \sum_{j \in m_i} E_{ij} B_i^{ij} R_{ij}^i + \sum_{j \in s_i} E_{ji} B_i^{ji} R_{ji}^i, \\ B &= \begin{pmatrix} B_1 & \cdots & B_N \end{pmatrix}. \end{aligned} \tag{2.8}$$

Then the mortar matching condition (2.7) becomes

$$Bw = 0. \tag{2.9}$$

Define

$$W_{ij}^0 := \{v \in W_{ij} \ : \ v = 0 \text{ on } \partial\Gamma_{ij}\}$$

and let

$$W_n = \prod_{i=1}^{N} \prod_{j \in m_i} W_{ij}^0. \tag{2.10}$$

We define by $E(w)$ the zero extension of the function $w \in W_n$ to the all interfaces, i.e., mortar and nonmortar interfaces. Hence for $w \in W_n$, we define a norm by

$$\|w\|_{W_n} := \|E(w)\|_W.$$

Let $\langle \cdot, \cdot \rangle_m$ be a duality pairing between M and W_n such that

$$\langle \lambda, w \rangle_m := \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij} \, ds \;\; \forall \, (\lambda, w) \in M \times W_n. \tag{2.11}$$

Using this, we define a dual norm on M by

$$\|\lambda\|_{(W_n)'} := \max_{w \in W_n \setminus \{0\}} \frac{\langle \lambda, w \rangle_m}{\|w\|_{W_n}}.$$

2.3. **FETI-DP operator.** In this section, we construct a FETI-DP operator for the problem (2.1) with the mortar matching condition as constraints. The derivation of the FETI-DP equation for the Lagrange multipliers follows [28]. However, the FETI-DP operator with mortar matching condition is different from Dryja and Widlund [11, 12] that eliminate unknowns on both interior and vertex nodal points and impose a mortar matching condition over W_r in (2.5) so that the resulting solution u does not satisfy the mortar matching condition (2.7). In [18], Kim and Lee eliminate only interior nodal points and impose the mortar matching condition on the function over W in (2.4).

For $w_i \in W_i$ we write

$$w_i = \begin{pmatrix} w_r^i \\ w_c^i \end{pmatrix},$$

where r and c stand for the nodal values on the edges and vertices. From now on, we use the subscripts I, r and c to represent the d.o.f. (degree of freedom) corresponding to nodes in the interior, on the edges and at the vertices, respectively.

Define W_c as the set of vectors which have d.o.f. corresponding to the union of subdomain vertices, that is, global corner points. For $w = (w_1, \dots, w_N) \in W$, since w is continuous at subdomain vertices, there exists $w_c \in W_c$ such that $L_c^i w_c = w_c^i$ for all $i = 1, \dots, N$, where the matrix L_c^i consists of 0 and 1 and restricts the value of w_c on the vertices of subdomain Ω_i.

Let g^i be the Schur complement forcing vector obtained from $\int_{\Omega_i} f v_i \, dx$. The Schur complement matrix S^i and vector g^i are ordered in the following way:

$$S^i = \begin{pmatrix} S_{rr}^i & S_{rc}^i \\ S_{cr}^i & S_{cc}^i \end{pmatrix}, \quad g^i = \begin{pmatrix} g_r^i \\ g_c^i \end{pmatrix}.$$

Let $B_{i,r}$ and $B_{i,c}$ be matrices that consist of the columns of B_i corresponding to the nodal points on the edges and at the vertices, respectively.

Then the problem (2.1) becomes: Find $(w_r, w_c, \lambda) \in W_r \times W_c \times M$ such that

$$S_{rr} w_r + S_{rc} w_c + B_r^t \lambda = g_r, \tag{2.12}$$

$$S_{cr} w_r + S_{cc} w_c + B_c^t \lambda = g_c, \tag{2.13}$$

$$B_r w_r + B_c w_c = 0, \tag{2.14}$$

where

$$
\begin{aligned}
&S_{rr} = \mathrm{diag}_{i=1,\dots,N}\left(S^i_{rr}\right), \quad S_{rc} = \begin{pmatrix} S^1_{rc}L^1_c \\ \vdots \\ S^N_{rc}L^N_c \end{pmatrix},\\
&S_{cr} = S^t_{rc}, \quad S_{cc} = \sum_{i=1}^N (L^i_c)^t S^i_{cc} L^i_c,\\
&B_r = (B_{1,r}, \dots, B_{N,r}), \;\; B_c = \sum_{i=1}^N B_{i,c} L^i_c,\\
&g_r = \begin{pmatrix} g^1_r \\ \vdots \\ g^N_r \end{pmatrix}, \; g_c = \sum_{i=1}^N (L^i_c)^t\ g^i_c, \; w_r = \begin{pmatrix} w^1_r \\ \vdots \\ w^N_r \end{pmatrix}.
\end{aligned}
\tag{2.15}
$$

Since S_{rr} is invertible, we solve (2.12) for w_r and substitute it into (2.14) and (2.13). Then we obtain

$$
\begin{pmatrix} F_{I_{rr}} & F_{I_{rc}} \\ F_{I_{cr}} & -F_{I_{cc}} \end{pmatrix} \begin{pmatrix} \lambda \\ w_c \end{pmatrix} = \begin{pmatrix} d_r \\ -d_c \end{pmatrix}.
$$

Eliminating w_c in the above equation, we obtain

$$
\left(F_{I_{rr}} + F_{I_{rc}} F^{-1}_{I_{cc}} F_{I_{cr}}\right)\lambda = d_r - F_{I_{rc}} F^{-1}_{I_{cc}} d_c.
$$

Here, $F_{DP} = F_{I_{rr}} + F_{I_{rc}} F^{-1}_{I_{cc}} F_{I_{cr}}$ is called the FETI-DP operator for the problem (2.1).

2.4. **Preconditioner and condition number estimation.** We propose $\widehat{F}^{-1}_{DP}$, a preconditioner for F_{DP}, which is derived from the dual norm on the Lagrange multiplier space M in the following sense:

$$
\langle \widehat{F}_{DP}\lambda, \lambda\rangle = \|\lambda\|^2_{(W_n)'}.
$$

For the derivation of the matrix form of $\widehat{F}_{DP}$, refer to [18]. We call $\widehat{F}^{-1}_{DP}$ a Neumann–Dirichlet preconditioner because it solves local problems with Neumann boundary conditions on nonmortar interfaces and with a zero Dirichlet boundary condition on mortar interfaces.

We obtain the following estimate [18].

Theorem 2.1. *For $\lambda \in M$, we have*

$$
\langle \widehat{F}_{DP}\lambda, \lambda\rangle \;\le\; \langle F_{DP}\lambda, \lambda\rangle \;\le\; C \max_{i=1,\dots,N} \left\{ \left(1 + \log\frac{H_i}{h_i}\right)^2 \right\} \langle \widehat{F}_{DP}\lambda, \lambda\rangle,
$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of H_i and h_i.

Corollary 2.1. *We have the following condition number estimate:*

$$
\kappa\left(\widehat{F}^{-1}_{DP} F_{DP}\right) \le C \max_{i=1,\dots,N} \left\{ \left(1 + \log\frac{H_i}{h_i}\right)^2 \right\},
$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of H_i and h_i.

3. Three-dimensional elliptic problem

3.1. **A model problem and FETI-DP operator.** We consider a FETI-DP method on nonmatching grids for the problem (2.1), where Ω is a bounded polyhedral domain in $\mathbb{R}^3$. Let Ω be partitioned into nonoverlapping polyhedral subdomains $\{\Omega_i\}_{i=1}^N$ similarly to the two-dimensional elliptic problem.

For $3D$ elliptic problems, it was shown from the numerical results in [13, 14] that using the primal variables at corners is not enough to get the same condition number bound as $2D$ problems. Hence, redundant continuity constraints are added to the coarse problem to accelerate the convergence of the FETI-DP method.

For the $3D$ elliptic problems with conforming discretizations, Klawonn *et al.* [23] developed FETI-DP methods with various redundant constraints. They introduced additional continuity constraints on edges or on faces to achieve the same condition number bound as $2D$ elliptic problems. The continuity constraints on edges are that the averages of functions across a common edge are the same. The same is applied to faces also. In [24], they extended the results to a case with face constraints only. However, only the face constraints

$$\int_{\Gamma_{ij}} v_i \, ds = \int_{\Gamma_{ij}} v_j \, ds \quad \forall i = 1, \cdots, N, \ j \in m_i. \tag{3.1}$$

are imposed in [16] as the redundant constraints since the constraints on edges are not redundant to the mortar matching condition. From $1 \in M_{ij}$, the above constraints are redundant to the mortar constraints (2.7) and they are written into the following algebraic equations:

$$R^t B w = 0, \tag{3.2}$$

where the matrix R has 0 or 1 as its entries and $R^t \lambda = 0$ means that sum of $\lambda|_{\Gamma_{ij}}$ is zero on each interface Γ_{ij}.

Let U be a Lagrange multiplier space corresponding to the redundant constraints (3.2). Then, we have the following mixed formulation of the problem (2.1) with the constraints (2.9) and (3.2): Find $(w_r, w_c, \mu, \lambda) \in W_r \times W_c \times U \times M$ satisfying

$$\begin{aligned} S_{rr} w_r + S_{rc} w_c + B_r^t R \mu + B_r^t \lambda &= g_r, \\ S_{cr} w_r + S_{cc} w_c + B_c^t R \mu + B_c^t \lambda &= g_c, \\ R^t B_r w_r + R^t B_c w_c &= 0, \\ B_r w_r + B_c w_c &= 0. \end{aligned} \tag{3.3}$$

In the above equations, we regard $\widetilde{w}_c = \begin{pmatrix} w_c \\ \mu \end{pmatrix}$ as primal variables in the FETI-DP formulation and follow the augmented FETI-DP formulation introduced in [14]. Then we have

$$\begin{pmatrix} K_{rr} & K_{rc} & B_r^t \\ K_{cr} & K_{cc} & \widetilde{B}_c^t \\ B_r & \widetilde{B}_c & 0 \end{pmatrix} \begin{pmatrix} w_r \\ \widetilde{w}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} g_r \\ \widetilde{g}_c \\ 0 \end{pmatrix}. \tag{3.4}$$

Since K_{rr} is invertible, after eliminating w_r in (3.4), we obtain

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{lc} & F_{ll} \end{pmatrix} \begin{pmatrix} \widetilde{w}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} -d_c \\ d_l \end{pmatrix}.$$

From the fact that $B_r^t R$ has a full column rank, we can show that F_{cc} is invertible. Hence, eliminating $\widetilde{w}_c$ in the above equation, the FETI-DP equation of (3.3) follows:

$$F_{DP}\lambda = d_l - F_{lc}F_{cc}^{-1}d_c, \tag{3.5}$$

with $F_{DP} = F_{ll} + F_{lc}F_{cc}^{-1}F_{cl}$. we call F_{DP} the FETI-DP operator. Since we added the redundant mortar matching constraints to the FETI-DP formulation, the solution of FETI-DP equation is not uniquely determined in M. Let us define a subspace

$$\widetilde{M} := \left\{ \lambda \in M \ : \ R^t\lambda = 0 \right\}. \tag{3.6}$$

In Section 3.2, we will show that F_{DP} is symmetric and positive definite (s.p.d.) on $\widetilde{M}$. Hence, the solution $\lambda \in \widetilde{M}$ is uniquely determined.

3.2. **Preconditioner and condition number estimation.** Since F_{DP} is s.p.d. on $\widetilde{M}$, we will solve (3.5) by the preconditioned conjugate gradient method using a suitable preconditioner. We derive a preconditioner from the similar idea to [18], in which a Neumann-Dirichlet preconditioner is derived from a dual norm on the Lagrange multiplier space by using a duality pairing between the Lagrange multiplier space and finite elements on nonmortar sides.

Let us define the following subspace equipped with a norm induced from W_n:

$$W_R^0 := \left\{ w \in W_n \ : \ R^t BE(w) = 0 \right\}.$$

A duality pairing between the spaces $\widetilde{M}$ and W_R^0 is defined as (2.11). Then, a dual norm on $\lambda \in \widetilde{M}$ is given by

$$\|\lambda\|_{\widetilde{M}} := \max_{w \in W_R^0} \frac{\langle \lambda, w\rangle_m}{\|w\|_{W_n}}.$$

Similarly to the $2D$ problem, we will find an operator $\widehat{F}_{DP}$ which gives

$$\langle \widehat{F}_{DP}\lambda, \lambda\rangle = \|\lambda\|^2_{\widetilde{M}} \tag{3.7}$$

and propose $\widehat{F}_{DP}^{-1}$ as a preconditioner for the operator F_{DP}. In order to obtain a matrix form of the operator $\widehat{F}_{DP}^{-1}$, see [16].

Now, we consider the following elliptic problem with discontinuous constant coefficients:

$$\begin{aligned} -\nabla \cdot (\alpha(x)\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.8}$$

with $\alpha(x)|_{\Omega_i} = \rho_i (> 0)$ for all $i = 1, \cdots, N$. At this point, we need a reasonable assumption on the ratio of meshes for $3D$ problems.

Assumption on meshes: For each Γ_{ij}, we assume that

$$\frac{h_j}{h_i} \leq C \left(\frac{\rho_j}{\rho_i} \right)^{\gamma}, \text{ with } 0 \leq \gamma \leq 1, \tag{3.9}$$

where the constant C does not depend on mesh parameters h_i, H_i, and coefficients ρ_i.

Now, we restrict ourselves to the elliptic problem (2.1) with coefficients $A(x)$ and $\beta(x)$ that do not change rapidly across subdomain interfaces or the elliptic problem (3.8) with discontinuous constant coefficients ρ_i's. Then we have the following result [16].

Theorem 3.1. *For $\lambda \in \widetilde{M}$, we have*

$$\|\lambda\|^2_{\widetilde{M}} \leq \langle F_{DP}\lambda, \lambda \rangle \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} \|\lambda\|^2_{\widetilde{M}},$$

where the constant C does not depend on mesh parameters H_i and h_i but may depend on coefficients $A(x)$ and $\beta(x)$ for the elliptic problem (2.1)*. For the elliptic problem* (3.8) *with discontinuous coefficients ρ_i's, the constant C is independent of mesh parameters and the coefficients.*

From (3.7) and the above theorem, we obtain the condition number bound.

Corollary 3.1. *For the elliptic problems* (2.1) *or* (3.8)*, we have*

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\},$$

where the constant C is the same as one in the above theorem.

4. Two-dimensional Stokes problem

4.1. **A model problem.** Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and the space $L^2_0(\Omega)$ contains functions in $L^2(\Omega)$ with zero average $\int_\Omega v\,dx = 0$. The space $H^1_0(\Omega)$ is a subspace of $H^1(\Omega)$ with functions having zero trace on the boundary of Ω.

In this section, we consider the following Stokes problem: For $\mathbf{f} \in [L^2(\Omega)]^2$, find $(\mathbf{u}, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega)$ satisfying

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ -\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= \mathbf{0} \text{ on } \partial\Omega. \end{aligned} \tag{4.10}$$

We then partition Ω into non-overlapping subdomains $\{\Omega_i\}_{i=1}^N$ as in Section 2.1. For each subdomain, we introduce the space $L^2_0(\Omega_i)$ and Π_0:

$$\begin{aligned} L^2_0(\Omega_i) &:= \left\{ q \in L^2(\Omega_i) \ : \ \int_{\Omega_i} q\,dx = 0 \right\}, \\ \Pi_0 &:= \left\{ q_0 \ : \ q_0|_{\Omega_i} \text{ is constant and } \int_\Omega q_0\,dx = 0 \right\}. \end{aligned} \tag{4.11}$$

The problem (4.10) is then written into an equivalent variational form:

Find $(\mathbf{u}, p_I, p_0) \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2 \times \prod_{i=1}^N L_0^2(\Omega_i) \times \Pi_0$ such that

$$\begin{aligned} \sum_{i=1}^N (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i} - \sum_{i=1}^N (p_I + p_0, \nabla \cdot \mathbf{v})_{\Omega_i} &= \sum_{i=1}^N (\mathbf{f}, \mathbf{v})_{\Omega_i} \quad \forall\, \mathbf{v} \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2, \\ -\sum_{i=1}^N (\nabla \cdot \mathbf{u}, q_I)_{\Omega_i} &= 0 \quad \forall\, q_I \in \prod_{i=1}^N L_0^2(\Omega_i), \\ -\sum_{i=1}^N (\nabla \cdot \mathbf{u}, q_0)_{\Omega_i} &= 0 \quad \forall\, q_0 \in \Pi_0, \\ \mathbf{u}|_{\Omega_i} - \mathbf{u}|_{\Omega_j} &= 0 \quad \forall\, \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j. \end{aligned} \tag{4.12}$$

Here $(\cdot,\cdot)_{\Omega_i}$ denotes the inner product in $[L^2(\Omega_i)]^n$ for $n = 1, 2$. Triangulations $\Omega_i^{2h_i}$ and $\Omega_i^{h_i}$ for pressure and velocity, respectively, are given in each subdomain. The finer triangulation $\Omega_i^{h_i}$ is obtained from $\Omega_i^{2h_i}$ by connecting mid points of edges in the triangle $\tau \in \Omega_i^{2h_i}$ so that τ is divided into four triangles. We assume that these triangulations are quasi-uniform and may not match across subdomain interfaces. The finite element space $P_1(h_i) - P_0(2h_i)$ is then associated with each subdomain Ω_i:

$$\begin{aligned} X_i &:= \left\{ \mathbf{v}_i \in \left[H_D^1(\Omega_i) \cap C(\Omega_i)\right]^2 \;:\; \mathbf{v}_i|_\tau \text{ is piecewise linear } \forall\, \tau \in \Omega_i^{h_i} \right\}, \\ Q_i &:= \left\{ q_i \in L_0^2(\Omega_i) \;:\; q_i|_\tau \text{ is constant } \forall\, \tau \in \Omega_i^{2h_i} \right\}. \end{aligned} \tag{4.13}$$

The inf-sup stability of the space $P_1(h_i) - P_0(2h_i)$ can be shown from the macro element technique in [32] or from the inf-sup stability of the space $P_2(2h_i) - P_0(2h_i)$ in [8]; see [19] for more details.

Our FETI-DP formulation will be described using the space X, an approximate space for velocity which can be discontinuous across the interfaces except corners, the space Q_I for pressure which has zero average in each subdomain, and the space W for velocity on the interfaces that is continuous at corners and can be discontinuous on the remaining part; the definitions of X and W are given in (2.2) and (2.4), respectively, except that v and w are replaced by vector functions $\mathbf{v}$ and $\mathbf{w}$, respectively:

$$Q_I := \prod_{i=1}^N Q_i \tag{4.14}$$

4.2. **Mortar methods.** We consider the space X for velocity and the space $P = Q_I \times \Pi_0$ for pressure to approximate the Stokes problem (4.12). We will impose the mortar matching condition on the velocity functions.

We take the Lagrange multiplier space M given in (2.6). Note that the function in M_{ij} is a vector function in this section. The mortar matching condition on

$\mathbf{v} = (\mathbf{v}_1, \cdots, \mathbf{v}_N) \in X$ is

$$\int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds = 0 \quad \forall \boldsymbol{\lambda}_{ij} \in M_{ij}, \forall\, i = 1, \cdots, N, \forall j \in m_i. \tag{4.15}$$

The mortar finite element space of velocity is then given by

$$V := \{\mathbf{v} \in X \;:\; \mathbf{v} \text{ satisfies the mortar matching condition (4.15)}\}.$$

It was shown in [5] that the best approximation property holds for the space $V \times P$, given by the Hood-Taylor finite element space for each subdomain, if it satisfies the inf-sup condition

$$\inf_{q \in P} \sup_{\mathbf{v} \in V} \frac{\int_\Omega \nabla \cdot \mathbf{v}\, q \, dx}{\|v\|^*_{1,\Omega} \|q\|_{0,\Omega}} \geq \beta,$$

where the constant β is independent of the mesh sizes and the subdomain sizes. Here $\|v\|^*_{1,\Omega}$ denotes the broken H^1-norm $\|v\|^*_{1,\Omega} = \left(\sum_{i=1}^N \|v\|^2_{1,\Omega_i} \right)^{1/2}$. To our best knowledge, there is no mathematical proof that shows the constant β is independent of the subdomain sizes. In [19] the inf-sup constant is computed numerically and observed that the constant is independent of the subdomain sizes as well as the mesh sizes. Therefore we can get as accurate a solution as in conforming approximations.

4.3. **FETI-DP formulation with primal constraints.** In this section, we formulate a FETI-DP operator with the mortar constraints (4.15). The unknowns $\mathbf{v}_i \in X_i$ and $\mathbf{w}_i \in W_i$ are ordered into

$$\mathbf{v}_i = \begin{pmatrix} \mathbf{v}_I^i \\ \mathbf{v}_r^i \\ \mathbf{v}_c^i \end{pmatrix}, \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_r^i \\ \mathbf{w}_c^i \end{pmatrix}.$$

We define the space X_I based on the splitting of unknowns:

$$X_I = \left\{ \mathbf{v}_I \;:\; \mathbf{v}_I|_{\Omega_i} = \mathbf{v}_I^i, \quad \forall i = 1, \cdots, N \right\}. \tag{4.16}$$

To solve the Stokes problem efficiently, we will consider the primal constraints (3.1) with vector functions $\mathbf{v}_i$ and $\mathbf{v}_j$. Note that this holds by replacing $\boldsymbol{\lambda}_{ij} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in (4.15) because constant multipliers belong to the Lagrange multiplier space M_{ij}. These primal constraints were introduced by Li [26, 27] to the Stokes problem with conforming discretization. The primal constraints enlarge the size of coarse problem so that it may lead to a fast convergence of the FETI-DP iteration.

We write (3.2) as

$$R^t (B_r \mathbf{w}_r + B_c \mathbf{w}_c) = \mathbf{0}. \tag{4.17}$$

Recall that B_r and B_c are defined in (2.15).

Now, we have the following mixed formulation of the problem (4.12):

Find $(\mathbf{u}_I, p_I, \mathbf{u}_r, \mathbf{u}_c, p_0, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in X_I \times Q_I \times W_r \times W_c \times \Pi_0 \times U \times M$ such that

$$\begin{pmatrix} A_{II} & G_{II} & A_{Ir} & A_{Ic} & G_{I0} & \mathbf{0} & \mathbf{0} \\ G_{II}^t & \mathbf{0} & G_{rI}^t & G_{cI}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{rI} & G_{rI} & A_{rr} & A_{rc} & G_{r0} & B_r^t R & B_r^t \\ A_{cI} & G_{cI} & A_{cr} & A_{cc} & G_{c0} & B_c^t R & B_c^t \\ G_{I0}^t & \mathbf{0} & G_{r0}^t & G_{c0}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R^t B_r & R^t B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_r & B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \\ \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{0} \\ \mathbf{f}_r \\ \mathbf{f}_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{4.18}$$

Here

$$\begin{pmatrix} A_{II} & A_{Ir} & A_{Ic} \\ A_{rI} & A_{rr} & A_{rc} \\ A_{cI} & A_{cr} & A_{cc} \end{pmatrix} \text{ is a stiffness matrix given by } \sum_{i=1}^{N} (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i},$$

$$\begin{pmatrix} G_{II}^t & G_{rI}^t & G_{cI}^t \end{pmatrix} \text{ is a matrix given by} \sum_{i=1}^{N} (-\nabla \cdot \mathbf{v}, p_I)_{\Omega_i},$$

$$\begin{pmatrix} G_{I0}^t & G_{r0}^t & G_{c0}^t \end{pmatrix} \text{ is a matrix given by} \sum_{i=1}^{N} (-\nabla \cdot \mathbf{v}, p_0)_{\Omega_i}.$$

The velocity spaces X_I, W_r and W_c are defined in (4.16), (2.5) and Section 2.3, respectively. For the definitions of spaces Q_I, Π_0 and M, see (4.14), (4.11) and (2.6), respectively. Since $p_0|_{\Omega_i}$ is constant and $\mathbf{v}_I|_{\partial\Omega_i} = \mathbf{0}$ for $\mathbf{v}_I \in X_I$, the divergence theorem gives $G_{I0} = \mathbf{0}$.

Let

$$\mathbf{z}_r = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_r \end{pmatrix}, \quad \mathbf{z}_c = \begin{pmatrix} \mathbf{u}_c \\ p_0 \\ \boldsymbol{\mu} \end{pmatrix}.$$

We regard $\mathbf{z}_c$ as a primal variable in the FETI-DP formulation and then write (4.18) into

$$\begin{pmatrix} K_{rr} & K_{rc} & \widetilde{B}_r^t \\ K_{rc}^t & K_{cc} & \widetilde{B}_c^t \\ \widetilde{B}_r & \widetilde{B}_c & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_r \\ \mathbf{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{f}}_r \\ \widetilde{\mathbf{f}}_c \\ \mathbf{0} \end{pmatrix}.$$

In Lemma 4.1 [19], it is shown that the matrix K_{rr} is invertible. After eliminating $\mathbf{z}_r$, we obtain the following equation for $\mathbf{z}_c$ and $\boldsymbol{\lambda}$:

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{cl}^t & F_{ll} \end{pmatrix} \begin{pmatrix} \mathbf{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}_c \\ \mathbf{d}_l \end{pmatrix}.$$

The matrix F_{cc}, a coarse problem in the FETI-DP formulation, is invertible with the assumption that the domain Ω has the triangulation to satisfy that

$$\begin{aligned} -\sum_i (p_0, \nabla \cdot \mathbf{v}_r)_{\Omega_i} + \sum_{i,j} \boldsymbol{\mu} \cdot \int_{\Gamma_{ij}} (\mathbf{v}_r^i - \mathbf{v}_r^j)\, ds &= 0 \quad \forall \mathbf{v}_r \in W_r \\ -\sum_i (p_0, \nabla \cdot \mathbf{v}_c)_{\Omega_i} + \sum_{i,j} \boldsymbol{\mu} \cdot \int_{\Gamma_{ij}} (\mathbf{v}_c^i - \mathbf{v}_c^j)\, ds &= 0 \quad \forall \mathbf{v}_c \in W_c \end{aligned}$$

give the solution $\begin{pmatrix} p_0 \\ \boldsymbol{\mu} \end{pmatrix} = 0$; see Lemma 4.2 [19].

Remark **4.1.** *Most triangulations of the domain Ω satisfy the above assumption because the number of velocity unknowns $\mathbf{v}_r$ and $\mathbf{v}_c$ is usually greater than the number of unknowns p_0 and $\boldsymbol{\mu}$.*

By eliminating $\mathbf{z}_c$, we then obtain the following equation for $\boldsymbol{\lambda}$:

$$F_{DP}\boldsymbol{\lambda} = \mathbf{d}_l - F_{cl}^t F_{cc}^{-1}\mathbf{d}_c, \tag{4.19}$$

where

$$F_{DP} = F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl}.$$

Since the primal constraints (4.17) are selected from the mortar matching condition, the solution $\boldsymbol{\lambda}$ is not uniquely determined in the space M. The matrix F_{DP} is s.p.d. on $\widetilde{M}$ given in (3.6); see Remark 4.3 [19]. Hence the solution $\boldsymbol{\lambda}$ of (4.19) is uniquely determined in $\widetilde{M}$.

4.4. **Preconditioner and condition number estimation.** In this section we will define finite element function spaces given on the interfaces and give the condition number bound of the FETI-DP operator with the Neumann-Dirichlet preconditioner.

For $\mathbf{w}_i \in W_i$, we define $S_i\mathbf{w}_i$ by

$$\begin{pmatrix} A_{II}^i & G_{II}^i & A_{Ir}^i & A_{Ic}^i \\ (G_{II}^i)^t & \mathbf{0} & (G_{rI}^i)^t & (G_{cI}^i)^t \\ A_{rI}^i & G_{rI}^i & A_{rr}^i & A_{rc}^i \\ A_{cI}^i & G_{cI}^i & A_{cr}^i & A_{cc}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \\ \mathbf{w}_r^i \\ \mathbf{w}_c^i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ S_i \begin{pmatrix} \mathbf{w}_r^i \\ \mathbf{w}_c^i \end{pmatrix} \end{pmatrix},$$

where the superscript i denotes the submatrix corresponding to the subdomain Ω_i. Since the upper left 2×2 matrix

$$\begin{pmatrix} A_{II}^i & G_{II}^i \\ (G_{II}^i)^t & \mathbf{0} \end{pmatrix}$$

represents the local Stokes problem with the Dirichlet boundary condition, it is invertible so that the Schur complement S_i is well-defined. We then assemble the local Schur complement matrices and define

$$S := \operatorname{diag}(S_1, \cdots, S_N). \tag{4.20}$$

It can be seen easily that S is s.p.d. on W.

We further define a subspace of W_n in (2.10), that satisfies a certain set of constraints:

$$\widetilde{W}_n := \left\{ \mathbf{w}_n \in W_n \ : \ \int_{\Gamma_{ij}} \mathbf{w}_n \, ds = 0, \quad \forall \Gamma_{ij} \right\}.$$

We now introduce the Neumann-Dirichlet preconditioner $\widehat{F}_{DP}^{-1}$ given by

$$\langle \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{W}_n \setminus \{0\}} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle}. \tag{4.21}$$

In [19], an explicit form of $\widehat{F}_{DP}^{-1}$ is provided in detail.

From [19], we have the condition number bound of our FETI-DP algorithm.

Theorem 4.1. *The FETI-DP algorithm with the Neumann-Dirichlet preconditioner* (4.21) *has the condition number bound*

$$\kappa(\widehat{F}_{DP}^{-1}F_{DP}) \leq C\frac{1}{\beta^2}\max_{i=1,\cdots,N}\left\{\left(1+\log\frac{H_i}{h_i}\right)^2\right\}.$$

5. Three-dimensional compressible elasticity

5.1. **A model problem.** Let Ω be a polyhedral domain in $\mathbb{R}^3$. We introduce the vector valued Sobolev spaces

$$\mathbf{H}_D^1(\Omega)=\prod_{i=1}^{3}H_D^1(\Omega),\quad \mathbf{H}^1(\Omega)=\prod_{i=1}^{3}H^1(\Omega)\,.$$

We then consider the elasticity problem: Find $\mathbf{u}\in\mathbf{H}_D^1(\Omega)$ such that

$$\int_\Omega G(\mathbf{x})\varepsilon(\mathbf{u}):\varepsilon(\mathbf{v})\,d\mathbf{x}+\int_\Omega G(\mathbf{x})\beta(\mathbf{x})(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\,d\mathbf{x}=\langle\mathbf{F},\mathbf{v}\rangle\quad\forall\mathbf{v}\in\mathbf{H}_D^1(\Omega), \tag{5.1}$$

where $G=E/(1+\nu)$ and $\beta=\nu/(1-2\nu)$ are material parameters depending on the Young's modulus $E>0$ and the Poisson ratio $\nu\in(0,1/2]$. We assume that ν is bounded away from $1/2$ so that we exclude the case of incompressible elasticity problems. The linearized strain tensor is defined by

$$\varepsilon_{ij}(\mathbf{u}):=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad i,j=1,2,3,$$

and the tensor product and the force term are given by

$$\varepsilon(\mathbf{u}):\varepsilon(\mathbf{v})=\sum_{i,j=1}^{3}\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}),\quad \langle\mathbf{F},\mathbf{v}\rangle=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}+\int_{\partial\Omega_N}\mathbf{g}\cdot\mathbf{v}ds.$$

Here $\mathbf{f}$ is the body force and $\mathbf{g}$ is the surface force on the natural boundary part $\partial\Omega_N$.

The space $\mathbf{ker}(\varepsilon)$ has the following six rigid body motions as its basis, which are three translations

$$\mathbf{r}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\ \mathbf{r}_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\ \mathbf{r}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{5.2}$$

and three rotations

$$\mathbf{r}_4=\frac{1}{H}\begin{pmatrix}x_2-\widehat{x}_2\\-x_1+\widehat{x}_1\\0\end{pmatrix},\ \mathbf{r}_5=\frac{1}{H}\begin{pmatrix}-x_3+\widehat{x}_3\\0\\x_1-\widehat{x}_1\end{pmatrix},\ \mathbf{r}_6=\frac{1}{H}\begin{pmatrix}0\\x_3-\widehat{x}_3\\-x_2+\widehat{x}_2\end{pmatrix}. \tag{5.3}$$

Here $\widehat{\mathbf{x}}=(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)\in\Omega$ and H is the diameter of Ω. This shift and the scaling make the L_2-norm of the six vectors scale in the same way with H. When Ω is partitioned into a set of subdomains, the elasticity problem given on a floating subdomain has purely natural boundary condition. The Korn inequalities provided in Section 2 of [22] concern this case.

5.2. **Domain decomposition with mortar discretization.** We divide the domain Ω into a geometrically conforming partition $\{\Omega_i\}_{i=1}^N$, that is shape regular. We consider a compressible elasticity problem with coefficients $G(\mathbf{x})$ and $\beta(\mathbf{x})$ that are positive constants in each subdomain

$$G(\mathbf{x})|_{\Omega_i} = G_i, \quad \beta(\mathbf{x})|_{\Omega_i} = \beta_i.$$

The conforming P_1-finite element space X_i and the trace space W_i of X_i are defined in (4.13) and (2.3), respectively with the superscript 3. The triangulations $\{\Omega_i^h\}_{i=1}^N$ may not match across the subdomain interfaces.

In the three dimensional case, a pair of subdomains can have a face, an edge, or a vertex in common. We will primarily consider only the common faces as the interfaces of subdomains. On each face $F^{ij} = \partial\Omega_i \cap \partial\Omega_j$, we will choose one of the two subdomains as the mortar side and the other as the nonmortar side depending on the coefficients $G(\mathbf{x})$, i.e., we will choose the subdomain with smaller $G(\mathbf{x})$ as the nonmortar side. We then introduce the finite element space

$$\mathbf{W}_{ij} = \left\{ \mathbf{w} \in \mathbf{H}_0^1(F^{ij}) \ : \ \mathbf{w} = \mathbf{v}|_{F^{ij}} \text{ for } \mathbf{v} \in X_{n(ij)} \right\},$$

where $n(ij)$ denotes the nonmortar side of F^{ij}. This space is spanned by a nodal basis $\{\boldsymbol{\phi}_k\}_{k=1}^{n_{ij}}$ related to the interior nodes of F^{ij} with respect to the triangulation $T^{n(ij)}$ of the nonmortar side. Based on this space, we construct a dual Lagrange multiplier space $\mathbf{M}_{ij}$ with a basis $\{\boldsymbol{\psi}_k\}_{k=1}^{n_{ij}}$ satisfying

$$\int_{F^{ij}} \boldsymbol{\phi}_l \cdot \boldsymbol{\psi}_k \, ds = \delta_{lk} \int_{F^{ij}} \boldsymbol{\phi}_l \, ds \quad \forall\, l, k = 1, \cdots, n_{ij}.$$

We refer to [15] for a detailed construction of the dual Lagrange multiplier space. The standard Lagrange multiplier space was introduced in [6] for three spatial dimensions. However the dual Lagrange multiplier space is more computationally efficient as well as easier to implement compared to the standard one. The mortar matching condition is then written as

$$\int_{F_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda} \, ds = 0 \quad \forall \boldsymbol{\lambda} \in \mathbf{M}_{ij}, \ \forall F_{ij}.$$

We then introduce the finite element spaces on the interfaces

$$\mathbf{W}_n = \prod_{i=1}^{N} \prod_{j \in m_i} \mathbf{W}_{ij}.$$

5.3. **Primal constraints in the FETI-DP formulation.** Solving linear systems arising from the mortar discretization is a difficult task [2, 3, 25]. Construction of the coarse finite element space in Schwarz-type algorithms or iterative substructuring algorithms, that provides scalability of the algorithms, is challenging in particular for three-dimensional problems with a geometrically non-conforming subdomain partition [1, 10]. On the other hand, the coarse problem in FETI-DP type algorithms follows from algebraic elimination of primal unknowns associated to the primal constraints. The selection of the primal constraints is important in achieving a scalable FETI-DP algorithm as well as in obtaining invertible subdomain problems.

Klawonn and Widlund [21] considered edge average and edge moment constraints, and vertex constraints for elasticity problems to control the rigid body motions of the subdomains as well as to obtain a scalable method. Furthermore they introduced the concepts of an acceptable face path and an acceptable vertex path in an attempt to reduce the number of primal constraints. Using constraints depending on edges is more promising than relying on faces when there are general distributions of jumps in the coefficients. Moreover the exchange of information between subdomains is related to a smaller set of unknowns. Numerical results support that edge constraints are more effective than face constraints [20].

For the case of mortar constraints, it is able to construct primal constraints based on faces. In [16], face average constraints are introduced for three dimensional elliptic problems with mortar discretizations and it was shown that the condition number is bounded by a polylogarithmic function of the subdomain problem size and is independent of the coefficients of elliptic problems.

We will now introduce six primal constraints on each face based on the idea in a recent study by Klawonn and Widlund [22]. On a face F^{ij}, we consider the rigid body motions $\{\mathbf{r}_i\}_{i=1}^6$ as in (5.2) and (5.3), where H is the diameter of the face F^{ij} and $\widehat{\mathbf{x}}$ is a point in F^{ij}. We define a projection $\mathbf{Q} : \mathbf{H}^{1/2}(F^{ij}) \to \mathbf{M}_{ij}$ by

$$\int_{F^{ij}} (\mathbf{Q}(\mathbf{w}) - \mathbf{w}) \cdot \boldsymbol{\phi} \, ds = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{W}_{ij}.$$

We then consider the projected rigid body motions $\{\mathbf{Q}(\mathbf{r}_i)\}_{i=1}^6$. Since the translational rigid body motions $\{\mathbf{r}_i\}_{i=1}^3$ are contained in $\mathbf{M}_{ij}$, $\mathbf{Q}(\mathbf{r}_i) = \mathbf{r}_i$ for $i = 1, 2, 3$. We now introduce the following constraints on the face F^{ij}

$$\int_{F^{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{Q}(\mathbf{r}_l) \, ds = 0 \quad \forall l = 1, \cdots, 6.$$

For $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=1}^3$, these constraints are nothing but the average matching conditions across the interface. The remaining constraints with $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=4}^6$, are similar to the moment matching constraints which were introduced for fully primal edges in [22] except that our constraints use the projected rotational rigid body motions and are imposed on faces. In the following, we call these constraints of $\{\mathbf{Q}(\mathbf{r}_l)\}_{l=4}^6$ the moment constraints.

Even though we have introduced the set of primal constraints in order to make the FETI-DP method more efficient, the enlarged coarse problem can be a bottle neck of the computation. To compromise between the number of iterations of the FETI-DP method and the size of coarse problem, we will not impose the primal constraints over all faces. Among the faces, we select some as primal faces and we impose the six constraints only over them. For the remaining (non-primal faces), we assume that they satisfy an acceptable face path condition. This assumption makes it possible for the FETI-DP method with primal faces to have a condition number bound comparable to when all faces are chosen to be primal. We now define an acceptable face path.

Definition 5.1. (Acceptable face path) *For a pair of subdomains (Ω_i, Ω_j) having the common face F^{ij} with $G_i \le G_j$, an acceptable face path is a path*

$$\{\Omega_i, \Omega_{k_1}, \cdots, \Omega_{k_n}, \Omega_j\}$$

from Ω_i to Ω_j such that the coefficient G_{k_l} of Ω_{k_l} satisfies the condition

$$TOL * (1 + \log(H_i/h_i))^{-1} (1 + \log(H_{k_l}/h_{k_l}))^2 * G_{k_l} \ge G_i$$

and the path from one subdomain to another is always through a primal face.

Some of faces are chosen as primal faces and the remaining are non-primal faces. In [17], an algorithm is introduced, which selects relatively few primal faces as well as keeps the condition number bound of the resulting FETI-DP operator within $C \max_{i=1,\cdots,N} \left\{(1 + \log(H_i/h_i))^2\right\}$. Here the constant C depends on the parameters TOL and L, the maximum number of subdomains on the acceptable face path. Furthermore, we choose some of vertices as primal vertices at which we will impose a point-wise matching condition. We assume that enough primal vertices are taken so as to make each local problem invertible. Based on these primal constraints, we introduce the following subspaces

$$\begin{aligned}\widetilde{\mathbf{W}} :&= \{\mathbf{w} \in W \ : \ \mathbf{w} \text{ satisfies the vertex constraints at the primal vertices} \\ &\qquad \text{and the six face constraints across each primal faces}\}, \\ \widetilde{\mathbf{W}}_n :&= \{\mathbf{w}_n \in \mathbf{W}_n \ : \ \mathbf{w}_n \text{ has zero average and zero moment} \\ &\qquad \text{on each primal faces}\}.\end{aligned}$$

For $\mathbf{w}_n \in \widetilde{\mathbf{W}}_n$, let $E(\mathbf{w}_n) \in W$ be the zero extension of $\mathbf{w}_n$ to the whole interface, i.e., mortar and nonmortar faces. We can easily see that $E(\mathbf{w}_n) \in \widetilde{\mathbf{W}}$.

5.4. **The FETI-DP equation and condition number analysis.** Let A_i denote the stiffness matrix of the bilinear form

$$a_i(\mathbf{u}_i, \mathbf{v}_i) := G_i \int_{\Omega_i} \varepsilon(\mathbf{u}_i) : \varepsilon(\mathbf{v}_i)\, dx + G_i \beta_i \int_{\Omega_i} (\nabla \cdot \mathbf{u}_i)(\nabla \cdot \mathbf{v}_i)\, dx,$$

and let S_i be the Schur complement of the matrix A_i. We note that we choose some of the vertices as primal vertices at which we will impose the point-wise matching condition. Let V_c be the set of unknowns at the global primal vertices and let $V_c^{(i)}$ be the set of unknowns at the primal vertices in the subdomain Ω_i. The mapping $R_c^{(i)} : V_c \to V_c^{(i)}$ is the restriction from the unknowns at the global primal vertices to the unknowns at the local primal vertices. The mortar matching matrix B_i and the vector $\mathbf{w}_i \in \mathbf{W}_i$ are ordered as

$$B_i = \begin{pmatrix} B_r^{(i)} & B_c^{(i)} \end{pmatrix}, \quad \mathbf{w}_i = \begin{pmatrix} \mathbf{w}_r^{(i)} \\ \mathbf{w}_c^{(i)} \end{pmatrix},$$

where c stands for the unknowns at the primal vertices and r stands for the remaining unknowns. We then assemble vectors and matrices from the subdomains

$$\mathbf{w}_r = \begin{pmatrix} \mathbf{w}_r^{(1)} \\ \vdots \\ \mathbf{w}_r^{(N)} \end{pmatrix}, \quad B_r = \begin{pmatrix} B_r^{(1)} & \dots & B_r^{(N)} \end{pmatrix}, \quad B_c = \sum_{i=1}^{N} B_c^{(i)} R_c^{(i)}.$$

The face constraints are selected from the mortar matching constraints and they can be written as

$$R^t(B_r \mathbf{w}_r + B_c \mathbf{w}_c) = 0,$$

where the matrix R gives the face constraints as linear combinations of rows of the matrix $\begin{pmatrix} B_r & B_c \end{pmatrix}$.

By introducing the Lagrange multipliers $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ for the primal face constraints and for the mortar matching constraints, respectively, we get the following mixed formulation of (5.1)

$$\begin{pmatrix} S_{rr} & S_{rc} & B_r^t R & B_r^t \\ S_{cr} & S_{cc} & B_c^t R & B_c^t \\ R^t B_r & R^t B_c & 0 & 0 \\ B_r & B_c & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w}_r \\ \mathbf{w}_c \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_r \\ \mathbf{g}_c \\ 0 \\ 0 \end{pmatrix}.$$

We now eliminate the unknowns other than $\boldsymbol{\lambda}$ and obtain

$$F_{DP} \boldsymbol{\lambda} = \boldsymbol{d}.$$

This matrix F_{DP} satisfies the well-known relation

$$\langle F_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w} \in \widetilde{\mathbf{W}}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle},$$

where B and S are given in (2.8) and (4.20), respectively.

We now introduce the Neumann-Dirichlet preconditioner M^{-1} given by

$$\langle M \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{\mathbf{W}}_n} \frac{\langle BE(\mathbf{w}_n), \boldsymbol{\lambda} \rangle^2}{\langle SE(\mathbf{w}_n), E(\mathbf{w}_n) \rangle},$$

where $E(\mathbf{w}_n)$ is the zero extension of $\mathbf{w}_n$ into the space $\mathbf{W}$. The explicit form of the preconditioner is given in [17].

We need the assumption (3.9) on the mesh sizes, for which ρ is replaced by G. Then we have the following condition number bound [17].

Theorem 5.1. *Suppose that every non-primal face satisfies the acceptable face path condition with a given TOL and L. Under the assumption for the mesh sizes, we obtain the condition number bound*

$$\kappa(M^{-1} F_{DP}) \leq C(TOL, L) \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\}.$$

Here the constant C is independent of the mesh parameters and the coefficients G_i, but depends on TOL and L, the maximum face path length.

Remark **5.2.** *For an algorithm which selects a quite small number of primal faces, see* [17].

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