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DIFFEOMORPHISMS WITH LIMIT SHADOWING PROPERTY

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ABSTRACT. In this article, we give a characterization of two-sided limit shadowing property for homeomorphism of S^1. Under C^1-robust point of view, we consider hyperbolicity of invariant closed subset of smooth manifold such as chain recurrent set and chain component via various limit shadowing properties and show some results for them.

0.1. **Limit Shadowing Property.** The notion of pseudo-orbits is very often appearing in several branches of the dynamical systems theory, and various types of the (pseudo-orbit) shadowing properties are presented and playing an important part in its investigation since Anosov and Bowen. Recently in [10], various shadowing properties such as limit shadowing proepty, average shadowing property, strong shadowing property, inverse shadowing property have been collected by Pilyugin and several developments for the properties are introduced and argued with the relation of hyperbolicity and stabiliity etc.

Among of these properties, Eirola et al. in [3] has suggested limit shadowing property induced by following motivation. In numerically approximating an orbit of a discrete dynamical system, one may take steps to increase the numerical accuracy of the approximation after one has encountered some interesting feature of the dynamics.i.e. the computational error made at each step may decrease as more and more steps are taken. This raises the question of whether there is an actual

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orbit shadowing the computed approximate orbit, and if so, whether the tail of the shadowing orbit can be expected to better approximate the tail of the computed orbit. The specific mathematical question is this:
if x_k is a sequence with $|x_{k+1} - f(x_k)| \to 0$, is there a point x^* such that $|x_k - f^k(x^*)| \to 0$, and can estimates on the size of terms in the former sequence by used to obtain estimates of the size of terms in the latter?

Consider a dynamical system generated by a homeomorphism f of a compact metric space (M, d). Eirola et al. in [3] has defined that f has the *one-sided limit shadowing property ($LmSP^+$)* if, for any sequence $\xi = \{x_k \in M : k \geq 0\}$ such that $d(f(x_k), x_{k+1}) \to 0$ as $k \to \infty$, there exists a point $p \in M$ such that $d(f^k(p), x_k) \to 0$, as $k \to \infty$. In general, we know less about the limit shadowing property($LmSP^+$) than about the usual shadowing property. For example, it is known that the POTP is C^0-generic in the space of homeomorphisms of any smooth closed manifold M [11], while no analog of this statement is known for the $LmSP^+$ if $dimM \geq 2$. Even though this, Eirola et al. in [3] has proved that a diffeomorphism has the $LmSP^+$ in a neighborhood of its hyperbolic set.

Let Λ be a subset of M. Λ *is* $LmSP^+$ for f if for any sequence $\xi = \{x_k | k \geq 0\} \subset \Lambda$ of f such that $d(f(x_k), x_{k+1}) \to 0,\ k \to \infty$, there is $p \in M$ satisfying $d(f^k(p), x_k) \to 0,\ k \to \infty$. Pilyugin in [9] has considered $LmSP^+$ globally in the C^1-robust point of view. He has defined that M *is C^1-stably* $LmSP^+$ for f if there exists a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, M is $LmSP^+$ for g. Note that M is stably $LmSP^+$ for f means that $f \in Int^1(LmSP^+)$, where $Int^1(LmSP^+)$ is the set of all diffeomorphisms having the $LmSP^+$.

Comparing with this property, the shadowing property for an invariant set has been studied in the C^1-robust point of view.

Theorem 0.1. *Some results about hyperbolicity via C^1-stable shadowability are as follows.*

- *M is C^1-stably shadowable for f $\Leftrightarrow$ Axiom A and strong transversality condition* [14].
- *$CR(f)$ is C^1-stably shadowable for f $\Leftrightarrow$ $CR(f)$ is hyperbolic for f* [8].
- *$C_f(p)$ is C^1-stably shadowable for f $\Leftrightarrow$ $C_f(p)$ is hyperbolic for f* [15, 17].

In this line, under C^1-robust point of view, it may be a natural approach to consider hyperbolicity of invariant closed subsets of smooth manifold such as whole manifold, chain recurrent set and chain component via limit shadowing property.

Theorem 0.2. ([10], Theorem 2.1) *If Λ is hyperbolic for f then there exists a neighborhood U of Λ which is $LmSP^+$ for f.*

Theorem 0.3. ([9];Theorem 1) *M is C^1-stably $LmSP^+$ for f if and only if f is Ω-stable.*

Remark 0.4. *In general, C^1-stably shadowable and C^1-stably $LmSP^+$ do not coincide. More precisely, by Theorem 0.1 and Theorem 0.3, in the case of whole manifold,*

- *C^1-stably $LmSP^+ \nRightarrow C^1$-stably shadowable.*
- *C^1-stably shadowable(SS) $\Rightarrow C^1$-stably $LmSP^+(\Omega S)$.*

Theorem 0.5. *$CR(f)$ is C^1-stably $LmSP^+$ for f if and only if f is Ω-stable.*

0.2. **TSLmSP and *sl*-TSLmSP of invariant sets.** Eirola et al. in [3] has also defined that f has the *LmSP* if, for any sequence $\xi = \{x_k \in M : k \in \mathbb{Z}\}$ such that

$$d(f(x_k), x_{k+1}) \to 0, \; k \to \pm\infty \tag{1}$$

there exists a point $p \in M$ such that

$$d(f^k(p), x_k) \to 0, \; k \to \pm\infty. \tag{2}$$

In fact, we can easily see that LmSP is stronger than LmSP$^+$.

Example 0.6. [10] *Consider the identity map f on the circle S^1 with coordinate $x \in [0, 1)$. Then this system does not have the LmSP$^+$, LmSP and SP. Also let X be a compact metric space with $\sharp X \geq 2$. Then the identity map on X does not have the LmSP.*

But in case of SP, if the dimension of X is equal to zero, then the identity map on X has the SP [4].

Example 0.7. *If a homeomorphism f of X has two different fixed points in X such that one point is sink and another point is source, then the map f does not have the LmSP.*

In this Example 0.7, if there is no other fixed points except these kind of fixed points, then the map f has the SP by Plameneskaya's result [12]. Moreover she proved that if a homeomorphism f of S^1 has the SP, then f has the LmSP$^+$[12]. Therefore the map f has the LmSP$^+$.

Remark 0.8. [9] *As shown in Remark 2* [9] *and the above Example 0.7, in the global study of a dynamical system, it is unreasonable to consider a two-sided analog of the LmSP without restrictions on the values $d(f(x_k), x_{k+1})$.*

Let Λ be a subset of M. Pilyugin in [12, 9] has defined that Λ is *TSLmSP* for f if exists $\delta_0 > 0$ such that if any δ_0-p.t $\xi = \{x_k \in M : k \in \mathbb{Z}\} \subset \Lambda$ of f holds $d(f(x_k), x_{k+1}) \to 0,\ |k| \to \infty$, then exists $p \in X$ satisfying

$$d(f^k(p), x_k) \to 0,\ |k| \to \infty.$$

In case of $\Lambda = M$, f is said to have the *TSLmSP*. In fact, Pilyugin in [9] has already defined this definition in the name of "*f has the TSLmSP on* Λ".

Note 0.1. Let Λ_1 and Λ_2 be subsets of M where $\Lambda_2 \subset \Lambda_1$. Then it follows from the definition that if Λ_1 is TSLmSP for f, then Λ_2 is TSLmSP for f.

Example 0.9. *Consider the identity map f on the circle S^1 with coordinate $x \in [0, 1)$. Then the identity map f does not have the TSLmSP. In case of a non-trivial examples, if there exist at least one saddle type fixed point of the circle diffeomorphism, then the map also does not have TSLmSP.*

Note 0.2. By definition, LmSP on Λ implies TSLmSP on Λ. But the converse does not hold in general. In fact, every circle diffomorphisms f of S^1 which have only two fixed points such as sink and source type fixed points f has the SP and TSLmSP, but does not have the LmSP.

Pilyugin has obtained the TSLmSP from hyperbolicity. More precisely he has proved the following.

Proposition 0.10. [10] *A diffeomorphism has the TSLmSP in a neighborhood of a hyperbolic set.*

Also he has obtained following (see Lemma 5 in [9]) using by the SP and Ω-stability.

Proposition 0.11. [9] *A structurally stable diffeomorphism has the TSLmSP.*

Remark 0.12. *Ω-stability need not to have the TSLmSP. See the characterization theorem for TSLmSP of the class $\mathcal{P}$ which is appeared on Lemma 7 and Remark 3 in* [9].

Akin in [1] has introduced that f has the *two-sided s-limit shadowing property (s-TSLmSP)* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i | i \in \mathbb{Z}\}$ of f, there exists $y \in M$ satisfying $d(f^i(y), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$, and if in addition $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$, then $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Lee et al. [5] has proved that for an expansive homeomorphism f on M, f has the SP if and only if f has the s-TSLmSP. When we consider "TSLmSP", a limit-shadowing point exists globally so that we don't have any information for initial data in general. This kind of difficulty make us to choose a little modified concept. In detail, restricting considerable pseudo-orbit to limit δ-pseudo-orbit and giving a local information for location of a limit-shadowing point, we consider the following definition.

Definition 0.13 (Proposition 3.2 in [3])**.** *Let Λ be a subset of M. We say that Λ is sl-TSLmSP for $f \iff$ if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that if any δ-p.t ξ of f satisfies $d(f(x_k), x_{k+1}) \to 0$ as $k \to \pm\infty$, then $\exists\ y \in M$ satisfying $d(f^k(y), x_k) \to 0$ as $k \to \pm\infty$ such that $d(f^k(y), x_k) < \varepsilon$ for all $k \in \mathbb{Z}$. In case of $\Lambda = M$, f is said to have the sl-TSLmSP.*

In fact, this property has been already dealt in [3]. We can see that the *sl*-TSLmSP is independent of the metric which is compatible with the topology of X. Moreover, this property is conjugacy-invarinat and iteration-invariant. Proposition 3.2 in [3] says that a diffeomorphism has the *sl*-TSLmSP in a neighborhood of a hyperbolic set. They proved this statement using by Banach contraction mapping theorem. In Proposition 0.17, we give a much simpler proof.

Example 0.14. *Let (X, d) be a compact metric space with $\sharp X \geq \aleph_0$. Then the identity map id_X on X does not have the sl-TSLmSP. Especially consider the identity map id on the Cantor set X. Then id_X has the SP, but does not have the sl-TSLmSP. Moreover, as in the following two proposition 0.15, the shift homeomorphism on the Cantor set does have SP and sl-TSLmSP simultaneously. At this moment, on the smooth manifold, we dont's know what kind of relations exist between SP and the sl-TSLmSP.*

Proposition 0.15. *Let f be an expansive homeomorphism having the SP. Then f has the TSLmSP and sl-TSLmSP.*

It follows from the definition that *sl*-TSLmSP is a stronger condition than TSLmSP. But under additional conditions, these are equivalent. We obtain the following result.

Corollary 0.16. *Let f be an expansive homeomorphism having the SP. f has the sl-TSLmSP if and only if f has the TSLmSP.*

Theorem 0.17. *A diffeomorphism has the sl-TSLmSP in a neighborhood of a hyperbolic set.*

The following proposition can be proved in similar idea with Proposition 0.15.

Theorem 0.18. *A structurally stable diffeomorphism f of M has the sl-TSLmSP.*

This can give us to consider the converse problem under C^1-robust point of view. It means that C^1-stable TSLmSP may be sufficient condition for structural stability. In fact, we can see that this is true (see Theorem 0.25).

0.3. **TSLmSP on the circle maps.** Plameneskays in [12] has given characterizations for some circle maps via SP and $LmSP^+$. In this line, we can give a similar characterizations for orientation preserving homeomorphism via $TSLmSP$ as in the following.

Theorem 0.19. *A homeomorphism f preserving orientation and with nonempty set $Fix(\phi) \neq \emptyset$ has the $TSLmSP$ if and only if the set $Fix(\phi)$ is nowhere dense, $\Phi(t) - t$ changes sign on $[0, 1]$, and there is no saddle type fixed points.*

In case of $Fix(\phi) = \emptyset$, we don't know this characterization can be applied in general. But we can see that a rational rotation map f of S^1 dose not have the TSLmSP. Also it follows from Proposition 6.4 in [13] that an irrational rotation map f of S^1 defined by $f(x) = x + \alpha, \alpha \in \mathbb{Q}^c$ dose not have the TSLmSP.

Problem 0.1. Is there any homeomorphism having $TSLmSP$ on S^1 for which $Fix(\phi) = \emptyset$?

0.4. **C^1-non density of *sl*-TSLmSP.** In case of SP, many researches have been proved that the shadowing property is not C^1-generic. Especially Bonatti et al. in [2] has proved that the SP is not C^1-dense in $\text{Diff}^1(M^3)$. We have obtained the following using their ideas.

Theorem 0.20. *Let $f : M \to M$ be a transitive diffeomorphism with a strong partially hyperbolic splitting defined on a compact 3-manifold. Assume that f has two hyperbolic periodic points p and q such that $dim(W^s(p)) = 2$ and $dim(W^u(q)) = 1$. Then f does not have the sl-TSLmSP.*

By this theorem, we can see that the *sl*-TSLmSP is not C^1-dense in $\text{Diff}^1(M^3)$.

Corollary 0.21. *There is a C^1-dense open subset $\mathcal{O}$ of the set of non-Anosov, strong partially hyperbolic, and robustly transitive diffeomorphisms of a compact 3-manifold, such that every $f \in \mathcal{O}$ does not verify the sl-TSLmSP.*

0.5. **Stable *sl*-TSLmSP on closed manifold.** In this section, we want to consider how the stable *sl*-TSLmSP is closely related to some useful invariant sets such as chain recurrent sets including whole manifold M.

Definition 0.22 (Choi08). *M is C^1-stably TSLmSP(sl-TSLmSP) for f if there exists a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, M is TSLmSP(sl-TSLmSP) for g.*

Let $\mathcal{S}$ denote the set of structurally stable diffeomorphism of any smooth closed manifold and $Int^1(TSLmSP)$(resp. Int^1(*sl*-TSLmSP) denote the set of all diffeomorphism having the TSLmSP. In fact, $f \in Int^1(TSLmSP)$ means that M is C^1-stably TSLmSP(resp. *sl*-TSLmSP) for f. Sakai in [14] and Pilyugin in [9] have proved following proposition.

Proposition 0.23. [14, 9] $Int^1(SP) = \mathcal{S} \subsetneq Int^1(TSLmSP)$.

As we mentioned in Remark 0.12, Ω-stability need not to have the TSLmSP. Even though this, we don't know the relation between Ω-stability and Int^1(TSLmSP). But we can obtain the following result.

Theorem 0.24. *$Int^1(TSLmSP) \subsetneq \Omega\mathcal{S}$.*

If we restrict additional condition to TSLmSP so as to define *sl*-TSLmSP, we can obtain the following characterization on structural stability. This proposition says that under C^1-robust point of view, the concept of SP coincides with that of *sl*-TSLmSP.

Theorem 0.25. *Int^1(sl-TSLmSP) $= \mathcal{S}$. In other words, M is C^1-stably sl-TSLmSP for f if and only if f is structurally stable.*

In fact, we can use the same ideas appeared on the proof of Proposition B in [6] which confirm the construction of pseudo-orbit used in the process of the proof to be applied in the above Theorem definitely.

Definition 0.26 (Choi08). *$CR(f)$ is C^1-stably TSLmSP(sl-TSLmSP) for f if there exists a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, $CR(g)$ is TSLmSP(sl-TSLmSP) for g.*

Note that if M is C^1-stably TSLmSP for f, then $CR(f)$ is C^1-stably TSLmSP for f.

Theorem 0.27. *$CR(f)$ is C^1-stably TSLmSP(sl-TSLmSP) for f if and only if f is Ω-stable.*

0.6. **Stable *sl*-TSLmSP on the chain component.**

Definition 0.28 (Choi08). *We say that the chain component $C_f(p)$ is C^1-stably sl-TSLmSP for f if there exists a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, $C_g(p_g)$ is sl-TSLmSP for g, where p_g is the continuation of $p = p_f$.*

Example 0.29. *Consider a Smale's horseshoe map f on S^2 with a hyperbolic saddle fixed point p. Then $C_f(p) = H(p, f)$ is C^1-stably sl-TSLmSP for f.*

Proposition 0.30. *Let $C_f(p)$ be C^1-stably sl-TSLmSP for f. Then there exist a C^1-nbd $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $H_T(p, g) = H(p, g) = C_f(g)$.*

From this proposition, the chain component $C_f(p)$ is sure to have the dominated splitting under the condition of C^1-stably *sl*-TSLmSP for f.

Theorem 0.31. *Let $C_f(p)$ be C^1-stably sl-TSLmSP for f. Then there exist a C^1-nbd $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $C_g(p)$ admits a dominated splitting $E(g) \oplus F(g)$.*

As we mentioned previously, in case of C^1-stably shadowable chain component for f, the hyperbolicity of $C_f(p)$ is obtained as in the following.

Theorem 0.32. [15, 17] *$C_f(p)$ is C^1-stably shadowable chain components for f if and only if $C_f(p)$ is hyperbolic*

In this line, we want to consider the following problem.

Problem 0.2. $C_f(p)$ is C^1-stably *sl*-TSLmSP for f if and only if $C_f(p)$ is hyperbolic ?

References

[1] E. Akin, *The General Topology of Dynamical Systems*, Graduate Studies in Math. 1, Amer. Math. Soc., Providence, RI, 1993.

[2] C. Bonatti, L. Diaz and G. Turcat, *Pas de "shadowing lemma" pour des dynamiques partiellement hyperboliques. There is no shadowing lemma for partially hyperbolic dynamics)*, C.R.Acad. Sci. Paris Sér. I Math. **330** (2000), no. 7, 587-592.

[3] T. Eirola, O. Nevanlinna, and S. Pilyugin, *Limit shadowing property*, Numer. Funct. Anal. Optim. 18 (1997), no. 1-2, 75-92.

[4] Fujii, S., *Distal homeomorphisms with the pseudo orbit tracing property and total disconnectedness*, Preprint.

[5] K. Lee and K. Sakai, *Various shadowing properties and their equivalence*, Discrete Contin. Dyn. Syst. 13 (2005), no. 2, 533-540.

[6] K. Lee and K. Sakai, *Structural stability of vector fields with shadowing*, J. Differential Equations, **232** (2007), 303-313.

[7] R. Mañé, *An ergodic closing lemma*, Ann. of Math., **116** (1982), 503-540.

[8] K. Moriyasu, *The topological stability of diffeomorphisms*, Nagoya Math. J., **123** (1991), 91-102.

[9] S. Pilyugin, *Sets of dynamical systems with various limit shadowing properties*, J. Dynam. Differential Equations 19 (2007), no. 3, 747-775.

[10] S. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Mathematics, 1706. Springer-Verlag, Berlin, 1999. xviii+271 pp.

[11] S. Pilyugin, and O. Plamenevskaya, *Shadowing is generic*, Topo. Appl. 97 (1999), 253-266.

[12] O. B. Plamenevskaya, *Pseudo-orbit tracing property and limit shadowing property on a circle*, Vestnik St. Petersburg Univ. Math. 30 (1997), no. 1, 27-30.

[13] M. Pollicott and M. Yuri, *Dynamical systems and ergodic theory*, London Math. Soc. Student Texts **40**, Cambridge University Press, Cambridge, 1998.

[14] K. Sakai, *Pseudo orbit tracing property and strong transversality of diffeomorphisms on closed manifolds*, Osaka J. Math., **31** (1994), 373-386.

[15] K. Sakai, *C^1-stably shadowable chain components*, Ergod. Th. & Dynam. Sys., vol. **28** (2008), 987-1029.

[16] K. Sakai, and O. Tarakanov, *Limit weak shadowing property*, Far East J. Dyn. Syst. 7 (2005), no. 2, 137-144.

[17] X.Wen, S. Gan, and L. Wen, *C^1-stably shadowable chain components are hyperbolic*, Preprint(Feb, 2008).

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