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# GLOBAL ASYMPTOTIC STABILITY OF GENERALIZED LIENARD EQUATION

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ABSTRACT. In this article, by means of the analogism, we obtain the Liapunov function of global asymptotic stability of a class of nonlinear second order system. With some condition, we give the necessary and sufficient conditions of its global asymptotic stability, which covers several results in [1], [2], [3], [4] and [5].

Due to the importance of nonlinear second order systems in the study of mathematical theory and in the practical application, many scientists have paid close attention to the global asymptotic stability of the following Lienard equation

$$\text{(E1)} \qquad \frac{dx}{dt} = y - F(x), \qquad \frac{dy}{dt} = -g(x).$$

First, we list some assumptions on $F(x), g(x)$ and $G(x)$, where $G(x) = \int_0^x g(x)dx$.
(A1) $xg(x) > 0, \quad (x \neq 0)$;
(A2) $\lim_{|x|\to+\infty} G(x) = +\infty$;
(A3) $F'(x) > 0, \quad (x \neq 0)$;
(A4) $xF(x) > 0, \quad (x \neq 0)$;
(A5) $xF(x) \geq 0, \quad (x \neq 0)$;
(A6) $\lim_{|x|\to+\infty} |F(x)| = +\infty$;
(A7) $F(x)$ doesn't equal identically zero in any interval of $x$;
(A8) $F(x)$ and $F(-x)$ are all infinity.

Then, in [1], J. Lasalle and S. Lefschetz obtained the global asymptotic stability of the zero solution of the system (E1) either by $(A1), (A2)$ and $(A3)$ or by $(A1), (A3)$ and $(A6)$; in [2], Y. X. Qin, M. Q. Wang and L. Wang obtained also by $(A1), (A2)$ and $(A4)$; in [3], Z. S. Li and M. Q. Wang by $(A1), (A4)$ and $(A6)$; in [4], C. X. Qian obtained either by $(A1), (A2), (A5)$ and $(A7)$ or by $(A1), (A5), (A7)$ and $(A8)$. By the assumptions $(A1)$ and $(A5)$, in [5], H. Q. Li showed that the necessary and sufficient condition of global asymptotic stable of the zero solution of the system (E1) is

$$\lim_{|x|\to+\infty} (|F(x)| + G(x)) = +\infty.$$

In this paper, we consider more generalized nonlinear second order system

$$\text{(E2)} \qquad \frac{dx}{dt} = \phi(y) - F(x), \qquad \frac{dy}{dt} = -g(x).$$

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where

$$F(x)=\int_0^x f(x)dx,\quad G(x)=\int_0^x g(x)dx,\quad \Phi(y)=\int_0^y \phi(y)dy.$$

For the sake of convenience, we suppose that all the dealt functions are continuous and ensure the existence and uniqueness of the system.

When $\phi(y)=y$, we obtained Lienard equation (E1).

Let $x=x, z=\phi(y)-F(x)$, we obtain the equivalent system

$$\text{(E3)}\qquad \frac{dx}{dt}=z,\qquad \frac{dy}{dt}=-\phi'(y)g(x)-f(x)z.$$

If $\phi'(y)\geq 0$ and $\phi(y)$ doesn't equal identically zero in any interval of $y$, then $y=\phi^{-1}(z+F(x))$.

The following is the main theorem in this article.

**Theorem 1.** *Assume that*

(1) $xg(x)>0,\ (x\neq 0)$;

(2) $xF(x)\geq 0$ *and* $F(x)$ *doesn't equal identically zero in any interval of* $x$;

(3) $\phi(0)=0,\ 0\leq\phi'(y)\leq\alpha<+\infty$, *and* $\phi'(y)$ *doesn't equal identically zero in any interval of* $y$. *Then the necessary and sufficient condition of global asymptotic stability of the zero solution of the system* (E2) *is*

$$\text{(E4)}\qquad \overline{\lim_{|x|\to+\infty}}(|F(x)|+G(x))=+\infty.$$

*Proof.* First according to the condition (1)-(3), we know that $F(0)=g(0)=\phi(0)=0$, and the system (1.2) has zero solution. And according to the condition (3), we know that $\phi(y)$ is a strictly monotone increasing function, $\Phi(\pm\infty)=+\infty$.

$(\Leftarrow)$ : If (E4) holds, discussion can be divided to several cases.

When $x>0$, there are two cases:

(I) $G(x)$ is bounded, and $\overline{\lim}_{|x|\to+\infty}F(x)=+\infty$.

(II) $G(x)$ is unbounded, $G(x)$ is monotone increasing, and

$$\lim_{x\to+\infty}G(x)=+\infty.$$

When $x<0$, there are two cases:

(III) $G(x)$ is bounded, and $\overline{\lim}_{|x|\to-\infty}F(x)=-\infty$.

(IV) $G(x)$ is unbounded, $G(x)$ is monotone increasing, and

$$\lim_{x\to-\infty}G(x)=+\infty.$$

Hence, for $-\infty<x<+\infty$, we can discuss the following cases:

(i) (I) and (III) hold,

(ii) (I) and (IV) hold,

(iii) (II) and (III) hold,

(iv) (II) and (IV) hold.

By means of analogism, we construct positive definite Liapunov function:

$$V(x,y)=\Phi(y)+G(x),$$

and so

$$\frac{dV}{dt}\Big|_{(E2)} = -g(x)F(x) \leq 0.$$

Suppose $(x(t), y(t))$ is the solution of the system (E2) and entire positive semitrajectory in $\frac{dV}{dt}|_{(E2)} = 0$. If $x(t) \neq a$(constant), then there exists time variable $T$ such that $x(T) \neq 0$. Without loss of generality, we assume that $x(T) > 0$ (when $x(T) < 0$, the method is same).

According to continuity of $x(t)$, we know that there exists the interval $(T_1, T_2)$ such that in this interval $x(t) > 0$. Then for $t \in (T_1, T_2), x(t)$ is strictly monotone increasing function and range of $x(t)$ is an interval. According to the condition (1), we know that

$$g(x) \neq 0, \quad \text{and} \quad g(x(t))F(x(t)) \equiv 0, \quad \text{for } x \neq 0.$$

Hence, if $t \in (T_1, T_2)$, then we have $F(x(t)) \equiv 0$, this is contradict to the condition (2). Thus we have $x(t) \equiv a$.

From the first equality of the system (E2), we obtain that $\phi(y(t)) = F(a)$. Then we have $y(t) = \phi^{-1}(F(a)) \equiv b$ (in which $b$ is a constant). From the second equality of the system (E2), we know that $g(a) = 0$, and so $a = 0$. From the condition of function $F$ and function $\phi$, we know that $b = 0$. Then we have $x(t) = 0, y(t) = 0$. Hence, there is no entire nontrivial positive semitrajectory in set $\frac{dV}{dt}|_{(E2)} = 0$.

Following we prove that in cases (i), (ii) and (iii), solutions of the system (E2) are all bounded in positive sense. For this, we take Shimanov's domains:

$$D_r^{(1)} : V(x,y) < r,\ N_1(r) < x < N_2(r), \quad (r > 0, N_1(r) < 0, N_2(r) > 0)$$
$$D_r^{(2)} : V(x,y) < r,\ x < N_2(r), \quad (r > 0, N_2(r) > 0)$$
$$D_r^{(3)} : V(x,y) < r,\ N_1(r) < x, \quad (r > 0, N_1(r) < 0)$$

According to the method in [3], we know that for any point $P(x_0, y_0), r, -N_1(r)$, $N_2(r)$ can be taken so large that $P(x_0, y_0) \in D_r^{(i)}, i = 1, 2, 3$, and the positive semitrajectory passing $P(x_0, y_0)$ doesn't go out of $D_r^{(i)}$. Because $D_r^{(i)}$ is bounded, any positive semitrajectory of (E2) is bounded. In the case (4), $V(x, y)$ is an infinitely great positive definite function. In a word, in any case, the zero solution of the system (E2) is globally asymptotically stable.

$(\Rightarrow)$ : Suppose that the zero solution of the system (E2) is globally asymptotically stable. We suppose that (E4) doesn't hold when $x > 0$ (when $x < 0$, the method is same), then $\overline{\lim}_{|x| \to +\infty}(|F(x)| + G(x)) < +\infty$. Thus $F(x)$ and $G(x)$ are bounded in $[0, +\infty)$. Hence, $F(x) < M, G(x) < M$, for $x \in [0, +\infty)$ ($M$ is a constant).

Consider equivalent system (E3) of the system (E2). We prove that there exist the solution its positive semitrajectory doesn't tend to the zero.

Suppose that constant $\beta > 1$. Because $\int_0^{+\infty} g(x)dx < M$, we can take that $x_0 > 0$ is so large such that

$$\int_{x_0}^{+\infty} g(x)dx < \frac{1}{\alpha}.$$

Let $z_0 = \beta + M + 1$, we can prove that for the trajectory setting out from $(x_0, z_0)$, we have $z(t) \geq \beta$, for $t \geq 0$. If $z(t) > \beta$ doesn't hold, we can take $t_1 > 0$ which let $z(t_1) = \beta$ and $z(t) \geq \beta$ (in which $t \in [0, t_1]$). Hence $\frac{dx}{dt} = z(t) \geq \beta$ and $x(t)$ is monotone increasing, $x(t) > x_0, x(0) = x_0, z(0) = z_0$. Therefore for $t \in [0, t_1]$, we have

$$\frac{dz(t)}{dt} = \Big( - f(x(t)) - \frac{\phi'(y(t))g(x(t))}{z(t)} \Big) z(t),$$

where $y(t) = \phi^{-1}(z(t) + F(x(t)))$. Taking integral from 0 to $t_0$ from both sides and $dx(t) = z(t)dt$ is noted, we have

$$\begin{aligned} z(t_1) - z_0 &= - \int_{x_0}^{x(t_1)} f(x(t))dx(t) - \int_0^{t_1} \frac{\phi'(y(t))g(x(t))}{z(t)} z(t)dt \\ &> -M - \frac{\alpha}{\beta} \int_{x_0}^{x(t_1)} g(x)dx \\ &> -M - \frac{\alpha}{\beta} \int_{x_0}^{+\infty} g(x)dx \\ &> -M - \frac{1}{\beta}, \end{aligned}$$

which

$$\beta - (\beta + M + 1) > -M - \frac{1}{\beta},$$

i.e., $\beta < 1$. This is contradict to the taken $\beta$. Hence, we have $z(t) > \beta$ for all $t > 0$.

From $\frac{dx}{dt} = z(t)$, we know that $x(t) \to +\infty$ (when $t \to +\infty$). According to equivalence for the system (E2), $x(t) \to +\infty$ (when $t \to +\infty$) holds too. This means that there exists the solution whose positive semitrajectory doesn't tend to the zero in the system (E2). Hence the proof is complete. □

Obviously, the results of [1]-[5] are all special cases of our Theorem 1. Now we give an example.

**Example 2.** In system (E2), we let $\phi(y) = y + c \sin y$, which $|c| \leq 1$ is a constant.

$$f(x) = \begin{cases} -(x+n+1)^2(x+n+2)^2, & -(n+2) \leq x \leq -(n+1), \\ (x+n)^2(x+n+1)^2, & -(n+1) \leq x < -n, \\ (n+1)(x-n)^2(x-n-1)^2, & n \leq x < n+1, \\ -(n+1)(x-n-1)^2(x-n-2)^2, & n+1 \leq x < n+2, \end{cases}$$
$$(n = 0, 2, 4, \cdots)$$

$$g(x) = \begin{cases} x^3 e^{-x^2}, & x > 0, \\ x^3, & x \leq 0. \end{cases}$$

Then

$$\begin{aligned} \Phi(y) &= \frac{1}{2} y^2 + c(1 - \cos y), \\ G(x) &= \begin{cases} \frac{1}{2}(1 - e^{-x^2} - x^2 e^{-x^2}), & x > 0 \\ \frac{1}{4} x^4, & x \leq 0. \end{cases} \\ \phi'(y) &= 1 + c \cos y. \end{aligned}$$

Hence we have
(1) $xg(x) > 0$, for $x \neq 0$,
(2) Obviously, we know that $xF(x) = x\int_0^x f(x)dx \geq 0$. Furthermore,

$$F(x) = 0 \iff x = \pm n \ (n = 0, 2, 4, \cdots),$$

(3) $\phi(0) = 0, 0 \leq \phi'(y) \leq 2$ and $\phi'(y) = 0$ if and only if

$$c = 1, y = 2k\pi \pm \pi \ (k = 0, \pm 1, \pm 2, \cdots),$$

or

$$c = -1, y = 2k\pi \ (k = 0, \pm 1, \pm 2, \cdots).$$

(4) If we take $\{x_n\} = \{n+1, n = 0, 2, 4, \cdots\}$, then

$$\begin{aligned} F(x_n) &= F(n+1) \\ &= \int_0^{n+1} f(x)dx \\ &= \int_n^{n+1} (n+1)(x-n)^2(x-n-1)^2 dx \\ &= \frac{n+1}{30} \to +\infty \quad (n \to +\infty). \end{aligned}$$

Furthermore, $G(x) < \frac{1}{2}$, for $x > 0$. If $x < 0$, we know that

$$|F(x)| \leq \frac{1}{30}, \quad \text{and} \quad G(x) = \frac{1}{4}x^4 \to +\infty, \quad (x \to -\infty).$$

According to (1)-(4), we know that the above example satisfies the all sufficient conditions of Theorem 1. Hence the zero solution of above example is globally asymptotically stable.

When $c = 0$ in above example, we obtain the Lienard equation, and our theorem can judge its global asymptotic stability of the zero solution.

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