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THE TEMPERED STABLE MODELS

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ABSTRACT. In recent years the tempered stable processes have been popular models for asset returns because they can explain the observed reality of financial market in more accurate way than the Brownian motion model. Such processes are pure jump Levy processes of which the Levy density is obtained by multiplying the Levy density of the stable processes with appropriate tempering functions. In this article, we will focus on two different tempered stable models for asset returns: the exponentially tempered stable (ETS) model and the modified tempered stable (MTS) model. The first one is obtained by using a exponential tempering function and the second one is defined by using a Bessel tempering function.

We first briefly review some results on 6-parametric ETS process, and then introduce a new 6-parametric family of MTS processes, which contain VG processes as a special subclass. We secondly show that the class of MTS processes is totally disjoint with that of ETS processes, but share many nice structural and analytical properties with the ETS processes. We thirdly compare the MTS processes with the ETS processes in terms of the decay rate of jump sizes. We finally discuss that every pure jump processes with infinity activity (in particular MTS process) can be approximated as a jump-diffusion processes.

1. Introduction

The empirical studies have shown that the Black-Scholes (BS) model(1973) does not explain some stylized facts such as heavy tails and non-zero skewness of the asset returns distribution. To remedy these drawbacks, Mandelbrot (1963) proposed the use of α-stable distributions for asset returns. This yields a pure jump Levy process. Unfortunately, stable distributions have too heavy tails to model real financial data adequately. Madan et al (1991) introduced a non-stable Levy process, named the variance gamma(VG) process, as a generalization of the BS model by using a Gamma time change. Like an α-stable process, VG processes is a pure jump Levy process. In order to obtain more flexible model than the VG model, Carr et al (2002) introduced the CGMY model which allows to take account for finite / infinite activity, and finite / infinite variation of jumps. In recent years, the exponentially tempered stable(ETS) processes have been considerably studied in the literature, under a number of different names, including the "truncated Levy fight" [Koponen (1995)], the "KoBoL" process [Boyarchenko and Levendorskii (2000)], the "CGMY" process [Carr et al. (2002)], "tempering stable processes" [Rosinski (2007)], and "Dampened Power Law" [Wu (2006)] etc.

Recently, Kim, Rachev, Chung and Bianchi (2009) have proposed a new 4-parametric family of tempered stable processes, called modified tempered stable(MTS)processes, obtained by using more flexible tempering functions than exponential functions. Our new tempering functions will be indeed chosen as the mixture of gaussian functions with gamma distributions on positive real line. Such tempering functions will be positive real valued functions involving modified Bessel function of second kind, which will be termed as Bessel tempering functions. One of the advantages of such choice is that they include a shape parameter which is not fixed in advance, unlike the ETS processes. Based on this reason, the MTS model has a greater modelling freedom, which may lead to better fit compared to the ETS model. It turns out that the MTS processes are generalization of VG processes, and that the class of MTS processes is totally disjoint with that of ETS processes for $0 < \alpha < 2$, but shares many nice structural and analytical properties with the ETS processes. It is shown that the MTS processes have slower decay rate of jump sizes than ETS processes. We finally discuss that every pure jump processes with infinity activity, in particular MTS process, can be approximated as a jump-diffusion processes.

This article is organized as follows. In Section 2, we first provide an easier understanding of the basic structure of Levy process in an intuitive way. We indeed discuss how to construct Levy processes from Brownian motion and compound Poisson process. In Section 3. we review the results of 6-parametric exponentially tempered stable process. In Section 4 we introduce and discuss on a new 6-parametric family of MTS processes, which contain VG processes as a special subclass, and then compute the characteristic function of MTS distribution. In Section 5, we show that the class of the MTS processes is totally disjoint with that of ETS processes, but share many nice structural and analytical properties with the ETS processes. In Section 6 we discuss that the MTS process can be approximated as a jump-diffusion processes.

2. Infinitely divisible distributions and Levy processes

We first discuss infinitely divisible distributions and give some examples, and then discuss on the construction of Levy processes from Brownian motion and compound process.

Let X be a real valued random variable. The characteristic function of X (or its distribution P_X) is defined by $\phi_X(u) = E[e^{iuX}]$ (or $\int_{-\infty}^{\infty} e^{iux} P_X(dx)$). The distribution of a random variable is uniquely determined by its characteristic function.

The distribution P_X of a random variable X is called *infinitely divisible*, if for each $n \in \mathbb{N}$, there exists a sequence $(X_j^{(1/n)})_{j=1}^n$ of i.i.d. random variables such that $X \sim X_1^{1/n} + \cdots + X_n^{1/n}$. Equivalently, the distribution P_X of X is said to be infinitely divisible, if for each $n \in \mathbb{N}$, there exists another random variable $X^{(1/n)}$ such that $\phi_X(u) = (\phi_{X^{(1/n)}}(u))^n$. The normal distribution and the (compound) Poisson distribution and their independent sum are all well known examples of infinitely divisible distributions.

Theorem 2.1 (Levy- Khintchine formular)**.** The distribution of a random variable X is infinitely divisible if and only if its characteristic function is of the form

(2.1)
$$\phi_X(u) = E[e^{iuX}] = \exp\left\{ i\gamma u - \frac{1}{2}\sigma^2 ut + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{(|x|<1)})\nu(dx) \right\},$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and ν is a Borel measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty}(1 \wedge \alpha^2)\nu(dx) < \infty$. The triplet (γ, σ^2, ν) is called the *Levy triplet*, and the exponent in (2.1) is called the *Levy exponent.* Moreover, γ is called the *drift term*, σ^2 the *diffusion coefficient* and ν the *Levy measure.*

Remark 2.2.

(1) If $\int_{|x|<1} |x|\nu(dx) < \infty$, then $\gamma' = \gamma - \int_{|x|<1} x\nu(dx) \in \mathbb{R}$ and hence $\phi_X(u)$ can be written as
$$\phi_X(u) = \exp\left[i\gamma' u - \frac{1}{2}\sigma^2 u + \int_{-\infty}^{\infty} (e^{iux} - 1)\nu(dx) \right].$$

(2) If $\int_{|x|\geq 1} |x|\nu(dx) < \infty$, then $\gamma'' = \gamma + \int_{|x|\geq 1} x\nu(dx) \in \mathbb{R}$, and hence $\phi_X(u)$ can be written as
$$\phi_X(u) = \exp\left[i\gamma'' u - \frac{1}{2}\sigma^2 u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux)\nu(dx) \right].$$

A continuous-time process $(L_t)_{t\geq 0}$ is called a *Levy process*, if it has stationary, independent increments and is stochastically continuous. Brownian motion, (compound) Poisson process and their independent sum are all well known examples of Levy processes.

From the definition of a Levy process we see that for any $t \geq 0$, $L_t = \sum_{j=1}^{n} X_j^{(n)}$ where $X_j^{(n)} = L_{jt/n} - L_{(j-1)t/n}$, $j = 1, 2, \cdots, n$ are i.i.d. random variables. Thus every Levy process can be associated with an infinitely divisible distribution. The following theorem shows that the converse is also true, i.e. given any infinitely divisible distributed random variable X, we can construct a Levy process $(L_t)_{t\geq 0}$ such that $L_1 = X$.

Theorem 2.3 (Construction of Levy process)**.** Let X be a infinitely divisible distributed random variable with the Levy triplet (γ, σ^2, ν). Then there exists a Levy processes $(L_t)_{t\geq 0}$ such that for any $t \geq 0$, $L_t = \gamma t + \sigma B_t + C_t + M_t$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion, $C_t = \sum_{k=1}^{N_t(\lambda)} \xi_k$ with $\lambda = \nu((-1,1)^c)$, $\{\xi_n\} \sim \frac{1}{\lambda}\nu_{|(-1,1)^c}$, and M_t is the limit as $\epsilon \downarrow 0$ of $M_t^{(\epsilon)} = \sum_{k=1}^{N_t(\lambda_\epsilon)} \xi_k - t\int_{-\infty}^{\infty} x1_{\{\epsilon<|x|<1\}}\nu(dx)$, with $\lambda_\epsilon = \nu(\epsilon < |x| < 1)$, $\{\xi_n\} \sim \frac{1}{\lambda_\epsilon}\nu_{|(\epsilon<|x|\leq 1)}$. Moreover,

(2.2)
$$\begin{aligned}\psi_{L_t}(u) &= E[e^{iuL_t}] \\ &= \exp\left\{ t\left(u\gamma - \frac{u^2\sigma^2}{2} + \lambda\int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{(|x|<1)})\nu(dx) \right) \right\}, u \in \mathbb{R}\end{aligned}$$

Remark 2.4.

(1) The integrand in the exponent in (2.2) is $O(1)$ on $|x| \geq 1$ and $O(x^2)$ on $|x| < 1$. Hence the integral converges.

(2) $(C_t)_{t\geq 0}$, is a compound Poisson process (of big jumps), and $(M_t)_{t\geq 0}$ is a martingale (of small jumps).

(3) The Levy measure $\nu(dx)$ will be often of the form $k(x)dx$, and we then refer to k as the *Levy density*.

The following theorem shows that path behaviors of a Levy process can be determined by its Levy measure.

Theorem 2.5 (Activity and variation)**.** Let L be a pure jump Levy process with Levy triplet $(\gamma, 0, \nu)$. Then

(1) It has finite activity (i.e. almost all sample paths of L have a finite many jumps on every compact interval) if and only if $\nu(\mathbb{R}) < \infty$.
(2) It has finite variation (i.e. almost all sample paths of L have a finite variation) if and only if $\int_{-1}^{1} |x|\nu(dx) < \infty$.

Example 2.6. The following examples are infinity activity pure jump Levy processes which are obtained by using Theorem 2.3

(1) Stable Process

In 1963 Mandelbrot used the non-normal stable Levy process to explain behaviour of real market asset returns such as skewness and excess kurtosis more accurately.

An α-stable process is defined as a pure jump process with a Levy triplet $(\gamma, 0, k)$ given as $\gamma \in \mathbb{R}$ and

$$k(x) = \frac{C_-}{|x|^{1+\alpha}} 1_{(x<0)}(x) + \frac{C_+}{x^{1+\alpha}} 1_{(x>0)}(x), \tag{2.3}$$

where C_-, $C_+ \geq 0$, $0 < \alpha \leq 2$.

It is easily shown that every α-stable process has infinite activity.

(2) The Variance Gamma (VG) process (Madan et al (1991, 1998))

In 1991 Madan et al introduced a non-stable Levy process, named the VG process as a generalized BS model to explain more accurately the asset returns and option prices as well.

A *VG process* is defined as a (Gamma) time-changed Brownian motion with drift μ and volatility σ. More precisely, the VG process $X = (X_t)_{t\geq 0}$ is defined by $X_t = \mu G_t + \sigma B_{G_t}$ where $G = (G_t)_{t\geq 0}$ is a Gamma process generated by the Gamma distribution with mean 1 and variance ν. The density of the gamma distribution is given by $f(x; 1, \nu) = \frac{\nu^{-t/\nu}}{\Gamma(t/\nu)} x^{t/\nu - 1} e^{-x/\nu}$. The VG process is a pure jump Levy process having 3-parameters ν, σ and ν. Equivalently, the VG process can be defined by a pure jump Levy process with the Levy triplet $(\gamma, 0, k)$ given as $\gamma \in \mathbb{R}$ and

$$k_{VG}(x) = \frac{C_- e^{-\lambda_- |x|}}{|x|} 1_{(x<0)}(x) + \frac{C_+ e^{-\lambda_+ x}}{x} 1_{(x>0)}(x), \tag{2.4}$$

where C_-, C_+, λ_-, $\lambda_+ > 0$. It is shown that the VG process has infinity activity, but finite variation because $\int_{-1}^{1} |x| k_{VG}(x)dx < \infty$.

(3) The CGMY process(Carr et al 2002)

In order to obtain a more flexible process than the VG process, Carr et al (2002) added one additional parameter to the VG process which allows to have finite activity, infinity activity and infinite variation.

It is well-known that α-stable distributions have infinite α-th moments for any $\alpha \leq p$ since their Levy densities decay at polynomial rate. Tempering of stable jumps with exponential rate is a one of possible choices to ensure finite moments of all orders.

An $\alpha - CGMY$ *process* is defined as a pure jump Levy process with the Levy triplet $(\gamma, 0, k)$ given as $\gamma \in \mathbb{R}$ and

$$k^Y_{CGMY}(x) = \frac{Ce^{-G|x|}}{|x|^{1+Y}} 1_{(x<0)}(x) + \frac{Ce^{-Mx}}{x^{1+Y}} 1_{(x>0)}(x), \tag{2.5}$$

where C, G, $M > 0$ and $Y < 2$.

(4) Tempering Stable Process

In 2007, Rosinski introduced a Levy process as an extended version of the ETS process :

An α-*tempering stable* process is defined as a pure jump Levy process with the Levy triplet $(\gamma, 0, k)$ given as $\gamma \in \mathbb{R}$ and

$$k(x) = \frac{k_-(x)1_{(x<0)}(x)}{|x|^{1+\alpha}} + \frac{k_+(x)1_{(x>0)}}{x^{1+\alpha}}, \tag{2.6}$$

where $0 < \alpha < 2$ and k_- and k_+ completely monotonic functions on $(0, \infty)$ and $(-\infty, 0)$ respectively.

3. The Exponentially Tempered Stable (ETS) Model

The ETS model is a tempered stable processes obtained by using the negative exponential tempering functions. An *exponentially tempered stable* (ETS) process $X = (X_t)_{t\geq 0}$ is defined as a pure jump Levy process with the Levy triplet $(\gamma, 0, k^\alpha_{ETS})$ given as $\gamma \in \mathbb{R}$ and

$$k^\alpha_{ETS}(x) = \frac{C_- e^{-\lambda_-|x|}}{|x|^{1+\alpha}} 1_{(x<0)}(x) + \frac{C_+ e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{(x>0)}(x), \tag{3.1}$$

where $C_\pm$, $\lambda_\pm > 0$ and $\alpha < 2$.

This model was originally introduced by Koponen in [237] and was used in financial modelling by Carr et al [5] and Wu in [16]. The CGMY model is a version of this model studied in [5].

Remark 3.1. It is known that the integral $\int_{|x|\geq 1} x k^\alpha_{ETS}(x)dx$ exists. So, $m = \gamma + \int_{|x|\geq 1} x k^\alpha_{ETS}(x)dx$ exists and finite, and hence we will write the characteristic function of the ETS process $X = (X_t)_{t\geq 0}$ as

$$\phi_{X_t}(u) = \exp\left(t\left\{ium + \int_0^\infty (e^{iux} - 1 - iux)k_{ETS}(x)dx)\right\}\right). \tag{3.2}$$

In this case, we will use a notation $X \sim ETS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$.

The following theorem shows that the characteristic function of the ETS process can be computed explicitly.

Theorem 3.2. [6] Let $X \sim ETS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$. The characteristic function of X_1 is given by

(1) $$\phi_{ETS}(u, t;, C_\pm, \lambda_\pm, \alpha) = \exp t\Big\{ium + \Gamma(-\alpha)[C_+[(\lambda_+ - iu)^\alpha - \lambda_+^\alpha + iu\alpha\lambda_+^{\alpha-1}] + C_-[(\lambda_- + iu)^\alpha - \lambda_-^\alpha + iu\alpha\lambda_-^{\alpha-1}]\Big\} \quad (\alpha \neq 0, 1).$$

(2) $\phi_{ETS}(u,t;,C_\pm,\lambda_\pm,\alpha) = \exp t\Big\{iu(m+C_+-C_-)+C_+(\lambda_+-iu)\ln(1-\frac{iu}{\lambda_+})$

$+C_-(\lambda_-+iu)\ln(1+\frac{iu}{\lambda_-})\Big\} \quad (\alpha=1).$

(3) $\phi_{ETS}(u,t;,C_\pm,\lambda_\pm,\alpha) = \exp t\Big\{ium - C_+[\frac{iu}{\lambda_+}+\ln(1-\frac{iu}{\lambda_+})]$

$-C_-[-\frac{iu}{\lambda_-}+\ln(1+\frac{iu}{\lambda_-})]\Big\} \quad (\alpha=0).$

We easily see that the case $\alpha = 0$ yield the VG process.

Theorem 3.3. [5] The ETS process

(1) has an infinity activity for $0 < \alpha < 2$ and has a finite activity for $-1 < \alpha < 0$.
(2) has jumps of infinity variation for $1 < \alpha < 2$ and of finite variation for $0 < \alpha < 1$.

4. The Modified Tempered Stable (MTS) Model

We will define a parametric tempering function by using the mixture of squared exponential function with a gamma distribution.

(1) Mixtures of exponential functions.
Consider the following two parametric functions which captures 'power law' :

$$(1) \quad ce^{-|x|/v} \qquad\qquad (2) \quad ce^{-x^2/v}$$

where c is an overall scale parameter, v is a shape parameter, and x is a jump size.

A function f is said to be *completely monotonic* over $(0,\infty)$ if $(-1)^n f^{(n)}(x) \geq 0$, $0 < x < \infty$, $n = 0,\ 1,\ 2\ \ldots$. By Bernstein theorem, any completely monotonic f on $(0,\infty)$ can be expressed as a mixture of an exponential function with some positive measure μ on $(0,\infty)$:

$$f(x) = \int_0^\infty e^{-xt}d\mu(t), \quad 0 < x < \infty$$

The exponential functions are obviously completely monotonic. So the class of CGMY processes is a subclass of tempering stable processes introduced by Rosinski as a special case.

(2) The choice of tempering functions:
Consider a mixture of gaussian functions with a gamma distribution defined as

$$k_+(x) = \int_0^\infty ce^{-\frac{x^2}{2t}} g_{\nu,1/2}(t)dt, \quad c > 0,\ \nu > 0, \tag{4.1}$$

where

$$g_{\nu,\lambda}(t) = \begin{cases} \frac{\lambda}{\Gamma(\nu)}(\lambda t)^{\nu-1}e^{-\lambda t} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

By noting that $x^\nu K_\nu(x) = (\frac{1}{2})^{1-\nu}\int_0^\infty e^{-\frac{x^2}{4t}}e^{-t}t^{\nu-1}dt$, we have

$$k_+(x) \equiv \frac{c}{\Gamma(\nu)}\int_0^\infty e^{-\frac{x^2}{4u}}u^{\nu-1}e^{-u}du = c\frac{2^{1-\nu}}{\Gamma(\nu)}x^\nu K_\nu(x), \quad \nu > 0, \tag{4.2}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind of order ν.

Remark 4.1. Let $B_\nu(x) = (|x|)^\nu K_\nu(|x|)$, $\nu > 0$. Then $B_{1/2}(x) = \sqrt{\pi/2}e^{-|x|}$,

Proposition 4.2.

(1) B_ν is completely monotonic if and only if $\nu \leq 1/2$. Thus B_ν can't be completely monotonic for $\nu > 1/2$.

Proof: The proof can be done by using the fact that $B_{1/2}(x) = \sqrt{\pi/2}e^{-|x|}$ and the integral representations of $x^\nu K_\nu(x), x > 0$, which is given below:

$$x^\nu K_\nu(x) = \frac{\sqrt{\pi}2^\nu}{\Gamma(\frac{1}{2}-\nu)} \int_1^\infty e^{-xt} \frac{1}{(t^2-1)^{\nu+1/2}} dt, \quad \nu < 1/2 \tag{4.3}$$

We will adopt the class of functions of the form $C(|\lambda x|)^\nu K_\nu(\lambda|x|)$, $\lambda > 0$, $\nu > 0$ as tempering functions to dampen the stable jump rates because it includes the exponential family of functions of the form $Ce^{-\lambda x}$ which are completely monotonic.

(3) The modified tempered stable (MTS) process.

The MTS processes are a class of tempered stable processes whose Levy density is obtained by multiplying the Levy density of a stable process with the Bessel tempering functions defined above.

A *Modified tempered stable* (MTS) process $X = (X_t)_{t\geq 0}$ is defined as a pure jump Levy process with the Levy triplet $(\gamma, 0, k^\alpha_{MTS})$ given as $\gamma \in \mathbb{R}$ and

$$k^\alpha_{MTS}(x) = C\left(\frac{q_\alpha(\lambda_-|x|)}{|x|^{1+\alpha}} 1_{(x<0)}(x) + \frac{q_\alpha(\lambda_+ x)}{x^{1+\alpha}} 1_{(x>0)}(x)\right), \tag{4.4}$$

where $q_\alpha(x) = B_{\frac{1+\alpha}{2}}(x)$, $C > 0$, λ_-, $\lambda_+ > 0$, and $\alpha \in (-2, 2)$.

Remark 4.3. It is shown that $m = k + \int_{|x|\geq 1} x k^\alpha_{MTS}(x) ds$ exists and finite, and hence we will write the characteristic function of the MTS process with the Levy triplet $(\gamma, \sigma^2, k^\alpha_{MTS})$ as

$$\phi_{X_t}(u) = \exp\left(t\left\{ium + \int_0^\infty (e^{iux} - 1 - iux) k^\alpha_{MTS}(x)dx\right\}\right). \tag{4.5}$$

In this case, we will now on use a notation $X \sim MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$.

Examples (special cases of the MTS process)**.**

(1) α - stable distribution is a MTS distribution with no Bessel tempering, $\lambda_\pm = 0$.
(2) VG distribution is a special case of MTS distribution with $\alpha = 0$ since $q_0(x) = B_{1/2}(x) = \sqrt{\pi/2}e^{-|x|}$.

(4) Characteristic Function of $MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ with $\alpha \in (0,1) \cup (1,2)$.

The following theorem gives the characteristic function of $MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$.

Theorem 4.4. [9] Let $X \sim MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$. Then its characteristic function $\Phi_{X_1}(u)$ is given as :

(1) if $\alpha \neq 1$ and $\alpha \neq 0$, then

$$\phi_{X_1}(u; \alpha, C_\pm, \lambda_\pm, m) = \exp(ium + G_R(u; \alpha, C_\pm, \lambda_\pm) + G_I(u; \alpha, C_\pm, \lambda_\pm), \tag{4.6}$$

where

$$G_R(u; \alpha, C_\pm, \lambda_\pm) = 2^{-\frac{\alpha+3}{2}}\sqrt{\pi}\Gamma(-\alpha/2)\Big[C_+((\lambda_+^2 + u^2)^{\frac{\alpha}{2}} - \lambda_+^\alpha) + C_-((\lambda_-^2 + u^2)^{\frac{\alpha}{2}} - \lambda_-^\alpha)\Big],$$

and

$$\begin{aligned} G_I(u;\alpha,C_\pm,\lambda_\pm) = \;& iu2^{-\frac{(\alpha+1)}{2}}\Gamma(\frac{1-\alpha}{2}) \\ & \left[C_+\lambda_+^{\alpha-1}\left(F(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{-u^2}{\lambda_+^2})-1\right)\right. \\ & \left. -C_-\lambda_-^{\alpha-1}\left(F(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{-u^2}{\lambda_-^2})-1\right)\right] \end{aligned}$$

Here, F denotes a hyper-geometric function .

(2) If $\alpha = 0$, then

(4.7)

$$\phi_{X_1}(u;0,C_\pm,\lambda_\pm,m) = \exp(ium+\sqrt{\frac{\pi}{2}}\Big[-C_+(\ln(1-\frac{iu}{\lambda_+})-\frac{iu}{\lambda_+})-C_-(\ln((1+\frac{iu}{\lambda_-})+\frac{iu}{\lambda_-})\Big].$$

(5) Moments of $MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$ with $\alpha \in (0,1)\cup(1,2)$.

The cumulants of a random variable X are defined by

$$c_m(X) = \frac{1}{i^m}\frac{d^m}{du^m}\log\phi_X(u)|_{u=0}. \tag{4.8}$$

Theorem 4.5. [9] Let $X \sim MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$. Then the cumulants $c_m(X_1)$ is given as

$$c_n(X_1) = \begin{cases} m & ,n=1 \\ 2^{n-\frac{\alpha+3}{2}}\Gamma(\frac{n+1}{2})\Gamma(\frac{n-\alpha}{2})(C_+\lambda_t^{\alpha-n}+C_-(-1)^n\lambda_-^{\alpha-n}) & ,n\geq 2 \end{cases} \tag{4.9}$$

By using the above theorem, we can obtain the first four moments in the following theorem.

Theorem 4.6. [9] Let $X \sim MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)(\alpha \neq 1)$. Then the first four moments of X_1 are given as :

$$\text{mean} = c_1(X_1) = m.$$

$$\text{variance} = c_2(X_1) = 2^{-\frac{\alpha+1}{2}}\sqrt{\pi}\Gamma(1-\frac{\alpha}{2})(C_+\lambda_+^{\alpha-2}+C_-\lambda^{\alpha-2}).$$

$$\text{skewness} = \frac{c_3(X_1)}{c_2(X_1)^{3/2}} = \frac{2^{\frac{\alpha+9}{4}}\Gamma(\frac{3-\alpha}{2})(C_+\lambda_+^{\alpha-3}-C_-\lambda_-^{\alpha-3})}{\pi^{3/4}(\Gamma(1-\frac{\alpha}{2})(C_+\lambda_+^{\alpha-2}+C_-\lambda^{\alpha-2}))^{3/2}}.$$

$$\text{kurtosis} = \frac{c_4(X_1)}{c_2(X_1)^2} = \frac{3\cdot 2^{\frac{\alpha+3}{2}}\Gamma(2-\frac{\alpha}{2})(C_+\lambda_+^{\alpha-4}+C_-\lambda_-^{\alpha-4})}{\sqrt{\pi}[\Gamma(1-\frac{\alpha}{2})(C_+\lambda_+^{\alpha-2}+C_-\lambda^{\alpha-2})]^2}$$

The following theorem shows that the path structure of the MTS process is determined by the parameter α .

Theorem 4.7. [9] The MTS process

(1) has an infinity activity for $0 < \alpha < 2$, and has a finite activity for $\alpha < 0$.
(2) has jumps of infinity variation for $1 < \alpha < 2$, and of finite variation for $0 < \alpha < 1$.
(3) has a completely monotone Levy density for $\alpha > 0$ or $-2 < \alpha < 0$, since

$$B_{-\frac{(1+\alpha)}{2}}(x) = \frac{\sqrt{\pi}}{\Gamma(-\frac{\alpha}{2})2^{\frac{(1+\alpha)}{2}}}\int_1^\infty e^{-xt}(t^2-1)^{\alpha/2}dt.$$

Remark 4.8 (interpretation of parameters)**.** Let $X \sim MTS(\alpha, C_+, C_-, \lambda_+, \lambda_-, m)$.

(1) λ_+ and λ_- determine the tail behaviour of the Levy density.
(2) C_+ and C_- determine the overall level of activity of jumps.
(3) α determines the fine structure of sample paths, and the shape of tempering function as well.

5. Comparison with the CGMY process

In this section we will make a comparison of the MTS process to the CTS process.

Proposition 5.1.

(1) The MTS and CGMY processes both contain VG processes as a special case.
(2) The both of the processes are pure jump processes with the Levy density of the form
$$k(x) = C\left(\frac{T_\alpha(\lambda_-|x|)}{|x|^{1+\alpha}}1_{(x<0)}(x) + \frac{T_\alpha(\lambda_+x)}{x^{1+\alpha}}1_{(x>0)}(x)\right)dx.$$
where T_α is a tempering function.
(3) The Levy densities of the CGMY and MTS process are both a completely monotone function. We note that $B_{\frac{1+\alpha}{2}}(x)$ is not completely monotone for $\alpha > 0$. Thus the class of MTS processes is totally disjoint with that of CGMY processes for $\alpha > 0$, but share many nice structural and analytical properties with the ETS processes such as fine structure(Theorem 4.7).
(4) The MTS and CGMY processes both have semi-heavy tails but the MTS distribution has slower tail decay rates than the CGMY distribution : if $\alpha > 0$.
   (i) $\frac{T_\alpha(\lambda_\pm|x|)}{|x|^{1+\alpha}} \sim const\cdot|x|^{-(1+\alpha)}$ for $|x| \to 0$.
   (ii) $\frac{T_\alpha(\lambda_\pm|x|)}{|x|^{1+\alpha}} \sim const\cdot|x|^{-(1+\alpha/2)}e^{-\lambda_\pm|x|}$ for $|x| \to \infty$.

6. Approximation of small jumps by Brownian motion

In this section we show that every infinite activity pure jump Levy process can be approximated by a jump-diffusion process and we apply this to the MTS process.

Let $X = (X_t)_{t\geq 0}$ be an infinite activity pure jump Levy process with a Levy triplet $(a, 0, k)$. From the construction of pure jump Levy process, we know that X can be represented as
$$X_t = at + J_t + \lim_{\epsilon\to 0^-} M_t^\epsilon$$
where
$$J_t = \sum_{k=1}^{N_t(t\lambda)} \xi_k, \quad \lambda = \int_{|x|\geq 1} k(x)dx, \quad (\mathbb{P}\circ\xi_k^{-1})(\cdot) = \frac{1}{\lambda}\int_{(\cdot)\cap(|x|\geq 1)} k(x)dx$$
for all $k \in \mathbb{N}$ and
$$M_t^{(\epsilon)} = J_t^{(\epsilon)} - t\int_{-\infty}^{\infty} x1_{\{\epsilon<|x|<1\}}\nu(dx),$$
where $J_t^{(\epsilon)}$ is a compound Poisson process defined by
$$J_t^{(\epsilon)} \equiv \sum_{k=1}^{N_t(\lambda_\epsilon)} \xi_k, \quad \lambda_\epsilon = \int_{\epsilon<|x|<1} k(x)dx, \quad (\mathbb{P}\circ\xi_k^{-1})(\cdot) = \frac{1}{\lambda_\epsilon}\int_{(\cdot)\cap(\epsilon<|x|<1)} k(x)dx$$

for all $k \in \mathbb{N}$. Therefore a natural idea is to approximate X_t by $X_t = at + J_t + M_t^\epsilon$. Let $R_t^\epsilon = X_t - X_t^\epsilon$. Then $E[R_t^\epsilon] = 0$, and

$$Var[R_t^\epsilon] \equiv \sigma^2(\epsilon) = t \int_{|x|<\epsilon} x^2 k(x) dx.$$

If $\frac{\sigma(\epsilon)}{\epsilon} \to \infty$ as $\epsilon \downarrow 0$, then $\frac{R_t^\epsilon}{\sigma(\epsilon)} \to B_t$ in distribution as $\epsilon \downarrow 0$ for an independent Brownian motion $(B_t)_{t\geq 0}$. Thus we can simulate $(X_t)_{t\geq 0}$ by an approximated process:

$$Y_t^\epsilon = X_t^\epsilon + \sigma(\epsilon) B_t$$

for an independent Brownian motion $(B_t)_{t\geq 0}$. In other words, we may approximate the small jumps by an independent Brownian motion (see more detais, Asmussen & Rosinski (2001)).

Example 6.1. Let $(X_t)_{t\geq 0}$ be the MTS process defined via its Levy triplet $(m, 0, k_{MTS}^\alpha)$, where

$$k_{MTS}^\alpha(x) = C \frac{|x|^{\frac{1+\alpha}{2}} K_{\frac{1+\alpha}{2}}(\lambda_-|x|)}{|x|^{1+\alpha}} 1_{(x<0)}(x) + C \frac{x^{\frac{1+\alpha}{2}} K_{\frac{1+\alpha}{2}}(\lambda_+ x)}{x^{1+\alpha}} 1_{(x>0)}(x).$$

where $C > 0$, $\lambda_\pm > 0$ and $\alpha < 2$. Then we have

$$\sigma^2(\epsilon) = \int_{|x|<\epsilon} x^2 k_{MTS}^\alpha(x) dx < K \int_{|x<\epsilon|} |x|^{1-\alpha} dx = \frac{2K}{2-\alpha} \epsilon^{2-\alpha},$$

for some a constant $K > 0$. and for every given $\delta > 0$ and for all $\epsilon > 0$ small enough, the same quantity with C replaced by $C - \delta$ is a lower bound, so that $\frac{\sigma(\epsilon)}{\epsilon} \sim \sqrt{\frac{2K}{2-Y}} \epsilon^{-\alpha/2} \to \infty \Leftrightarrow \alpha > 0$. Hence an approximation of the small jumps of size $(-\epsilon, \epsilon)$ thrown away by a Brownian motion $\sigma(\epsilon) B_t$ is appropriate if and only if $\alpha > 0$.

7. Conclusion

We introduced a new tempered stable model for asset returns which we call the modified tempered stable(MTS) model. We made comparisons of the MTS model with the ETS model: we indeed showed that the class of the MTS processes is totally disjoint with that of ETS processes, but share many nice structural and analytical properties with the ETS processes. We showed that their characteristic functions are computationally tractable, and that the MTS processes can be approximated by a jump-diffusion processes. All these features make our process attractive for modelling of asset returns. The further work on a GARCH option pricing model with MTS innovation can be discussed (see details [9, 11])

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