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TRADING VOLATILITY AND HEDGING VOLATILITY RISK

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ABSTRACT. In this short survey note, we introduce several derivatives on volatility and explain how to hedge volatility risk. We also explain how to price and replicate volatility derivatives.

1. INTRODUCTION

When we apply Black-Scholes option pricing formula to the real financial markets, determining the volatility must be the most concerning task. In practice, volatility is defined by the annualized standard deviation of the log return [3]. Let $S_0, S_1, \cdots, S_n$ be the time series of daily stock prices, and let

$$u_i = \log \frac{S_i}{S_{i-1}},$$

for $i = 1, 2, \cdots, n$. Then

$$\sigma_{day} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \mu)^2},$$

where $\mu = \sum_{i=1}^{n} u_i/n$. Multiplying $\sqrt{252}$ to σ_{day}, we obtain the annualized volatility σ,

$$\sigma = \sqrt{252}\sigma_{day}.$$

The future volatility on any time horizon $[0, T]$ can be expressed by the same way and will be realized at time T. Trading this future volatility is now popular, which provides a hedge skim for volatility risk. In this short survey note, we introduce a couple of derivatives on volatility and explain how to hedge volatility risk. We also explain how to price volatility derivatives. For more details, see Carr and Madan [1], Demeterfi et. al. [2], and Bossu et. al. [4].

2. Volatility Swaps and Variance Swaps

For a given time horizon $[0, T]$, one can trade volatility through specifying the volatility realized in the future. Consider a future contract on volatility with delivery price K and the notional amount N, which gives the following payoff at T.

$$N \times (\sigma - K). \tag{1}$$

This contract is called volatility swap and it is common that σ and K are expressed in percent. For example, assume the notional amount N is 1 million dollar, maturity T is 3 months, and the delivery price is 20%. Suppose you long the volatility swap and after 3 months the realized volatility for 3 months becomes 30 %. Then you are paid

$$(30 - 20) \times 1 \ million = 10 \ million.$$

Therefore, if your are exposed to a volatility risk, you can hedge it through the above swap. Note that, since it is a future contract, if the volatility is realized below K, then there is a loss.

Practically, it is easy to make a replication if we consider the variance $V = \sigma^2$ instead of the volatility σ. Indeed, the variance swap is a contract which gives the following payoff.

$$N \times (V - K). \tag{2}$$

3. Pricing Variance Swap

To find the fair delivery price K of the variance swap, consider the martingale measure Q. Under Q, the expected payoff should be 0, in order for the variance swap to be a fair contract. That is,

$$E^Q(N \times (V - K) = 0.$$

Therefore, we have

$$K = E^Q(V).$$

That is, we can determine K by calculating $E^Q(V)$.

Let S_t be a stock price process. We assume S_t follows a diffusion process. Then under the martingale measure Q, with the constant risk free interest rate r, S_t can be expressed as

$$dS_t = rS_t dt + \sigma(t, S_t) S_t dB_t, \tag{3}$$

where B_t is a Brownian Motion and $\sigma(t, x)$ satisfies suitable growth conditions which guarantee the existence of the unique solution of the stochastic differential equation. Applying Ito's Lemma to $\log S_t$, we have

$$d(\log S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dS_t)^2 \tag{4}$$

$$= rdt + \sigma(t,S_t)dB_t - \frac{\sigma^2(t,S_t)}{2}dt \tag{5}$$

$$= (r - \frac{\sigma^2(t,S_t)}{2})dt + \sigma(t,S_t)dB_t. \tag{6}$$

¿From (3), we have

$$\frac{dS_t}{S_t} = rdt + \sigma(t,S_t)dB_t. \tag{7}$$

Then, subtract (4) from (7), we get

$$\frac{dS_t}{S_t} - d(\log S_t) = \frac{\sigma^2(t,S_t)}{2}dt. \tag{8}$$

For a given time horizon $[0,T]$, for some $T > 0$, let V be the annualized variance on $[0,T]$. Then

$$V = \frac{1}{T}\int_0^T \sigma^2(t,S_t)dt \tag{9}$$

$$= \frac{2}{T}\{\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\}. \tag{10}$$

Hence, the expected variance under Q is given by

$$E^Q(V) = \frac{2}{T}E^Q\{\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\} \tag{11}$$

$$= \frac{2}{T}\{E^Q(\int_0^T \frac{dS_t}{S_t}) - E^Q(\log\frac{S_T}{S_0})\}. \tag{12}$$

Here,

$$E^Q(\int_0^T \frac{dS_t}{S_t}) = E^Q(\int_0^T rdt + \int_0^T \sigma(t,S_t)dB_t) \tag{13}$$

$$= rT, \tag{14}$$

since $\int_0^T \sigma(t,S_t)dB_t$ is a martingale under Q.

Now for any $S^*(> 0)$, let

$$I = \int_0^{S^*} \frac{1}{K^2}(\max(K - S_T, 0))dK + \int_{S^*}^{\infty} \frac{1}{K^2}(\max(S_T - K, 0))dK, \tag{15}$$

and let us calculate I.

(i) For the case that $S_T \geq S^*$

$$I = \int_{S^*}^{S_T} \frac{1}{K^2}(S_T - K)dK \tag{16}$$

$$= \int_{S^*}^{S_T} (\frac{S_T}{K^2} - \frac{1}{K})dK \tag{17}$$

$$= -\frac{S_T}{K}|_{S^*}^{S_T} - \log K|_{S^*}^{S_T} \tag{18}$$

$$= -1 + \frac{S_T}{S^*} - \log\frac{S_T}{S^*}. \tag{19}$$

(ii) For the case that $S_T < S^*$

$$I = \int_{S_T}^{S^*} \frac{1}{K^2}(K - S_T)dK \tag{20}$$

$$= \int_{S^*}^{S_T} \frac{1}{K^2}(S_T - K)dK, \tag{21}$$

which is the same as (i).

Therefore, from (i) and (ii) we have

$$\log\frac{S_T}{S^*} = \frac{S_T}{S^*} - 1 - I. \tag{22}$$

Hence,

$$E^Q(\log\frac{S_T}{S_0}) = E^Q(\log\frac{S_T}{S^*} + \log\frac{S^*}{S_0}) \tag{23}$$

$$= E^Q(\frac{S_T}{S^*}) - 1 - E^Q(I) + \log\frac{S^*}{S_0}. \tag{24}$$

Here,

$$E^Q(S_T) = S_0 e^{rT} = F_0,$$

where F_0 is the future price of the stock. Now, $E^Q(I)$ can be calculated by exchanging the integral by Fubini Theorem. That is,

$$E^Q(I) = \int_0^{S^*} \frac{1}{K^2} E^Q(\max(K - S_T, 0))dK \tag{25}$$

$$+ \int_{S^*}^{\infty} \frac{1}{K^2} E^Q(\max(S_T - K, 0))dK \tag{26}$$

$$= e^{rT}(\int_0^{S^*} \frac{1}{K^2}P(K)dK + \int_{S^*}^{\infty} \frac{1}{K^2}C(K)dK), \tag{27}$$

where $P(K)$ and $C(K)$ are Black-Scholes put and call option prices, respectively. Therefore,

(28)

$$E^Q(V) = \frac{2}{T}[rT - \log\frac{S^*}{S_0} + 1 - \frac{F_0}{S_0} + e^{rT}(\int_0^{S^*} \frac{1}{K^2}P(K)dK + \int_{S^*}^{\infty} \frac{1}{K^2}C(K)dK)].$$

This equation implies that the variance swap can be replicated by put options with strike price less than S^* and call options with strike price greater than S^*. Indeed, Suppose options with strike prices

$$K_1 < K_2 < \cdots < K_{n_0} < s^* < K_{n_0+1} < \cdots < K_n,$$

are traded in the real market. Assume $K_i - K_{i-1} = \Delta K$, then we can discretize (28) by

$$E^Q(V) = \frac{2e^rT}{T}\left(\sum_{i=1}^{n_0} \frac{\Delta K}{K^2} P(K) + \sum_{i=n_0+1}^{n} \frac{1}{K^2} C(K)\right) - \frac{1}{T}\left(\frac{F_0}{S^*} - 1\right)^2. \tag{29}$$

Here, we can choose S^* near at the money, and we can successfully replicate the variance.

References

[1] Carr P, Madan D, 1998, Towards a Theory of Volatility Trading, in Volatility: New estimation Techniques for Pricing Derivatives. edited by R. Jarrow, 417-427.

[2] Demeterfi K, Derman E, Kamal K, and Zou J, 1999, More Than You Ever. Wanted To Know About Volatility Swaps, *Goldman Sachs Quantitative Strategies Research Notes*

[3] Hull J., 2008, Fundamentals of Futures and Options Markets, 6th edition, *Prentice Hall.*

[4] Bossu S, Strasser E, and Guichard R, 2005, Just What You Need to Know about Variance Swaps, *JP Morgan*.

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