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EFFICIENT MONTE CARLO METHOD FOR PATH-DEPENDENT EXOTICS

BYOUNG KI SEO

Abstract. A Monte Carlo method is one of the most frequently used methods to price financial exotic derivatives. It can be used for almost all financial derivatives easily except american style ones. Especially for path-dependent exotics, it can be a most useful method since we can easily give conditions to generated sample paths. However, generating sample paths with daily grids for giving conditions could waste the performance. Alternative solution using a probability density will be introduced and applied to pricing an ELS, one of the most exotic derivatives in Korea.

1. Introduction

A cheap money policy driven by the Lehman Brothers' bankrupcy is still going on in main financial markets of the world. Majorities of participants on financial market, including investors, traders and analysts had expected that a central bank of each country would start tighten the market from 2010. But the serial financial crisis from the Southern Europe and high unemplyment caused continuing low interest rate policies. Words of the 2010 for the Korea financial market was also "There is too much money." influenced by a low interest rate policy. These kinds of financial situations make investors pursue higher yield financial instruments. And structured products derived by a relatively safe market such as Korea financial market can be one of solutions.

The Korea derivatives market is one of the biggest market in the world. Actually, the KRX(Korea Exchange) is the biggest derivatives exchange with regard to the size traded a year, which means that it is the most liquid exchange in the world. Nevertheless, explaining trends of the structured note in Korea is not so difficult since most of them are composed by vanilla instruments such as interest rate swaps. However, one of the most exotic structured note in Korea would be equity linked one, which can be easily guessed since it is relatively simple to explain the structure to general investors and to model as a financial engineering one compare to something using interest rate 'curve'.

Trading Department, Industrial Bank of Korea, Seoul 100-758, Korea.

The ELS(Equity Linked Security) called "Two stock step down trigger" appeared around 2002 in Korea. Basic properties of this structure are "short vega" and "auto call", which means that investors can be betting into a volatility decreasing and it can be redeemed automatically before the whole life matured. The size of ELS issued in Korea during 2010 was 25 trillion Won and 67% of them were "Two stock step down trigger". Then how can structurers/traders price/manage it? Since that have been popular, lots of financial firms started to need so-called 'Quants'.

In this article we introduce widely used methods to price "Two stock step down trigger" and also introduce another method which is not so popular but helpful to someone have an interest in efficient Monte Carlo methods.

2. How to price the ELS

The ELS called "Two stock step down trigger" which can be named differently is specified as follows.

Two stock step down trigger (sample)

Trade date (T0): 201x-xx-xx
Seller: Party A
Buyer: Party B
Expiration Date (T6): T0 + 36 months
Underlying Shares: Underlying 1, Underlying 2
Notional Amount: KRW 10,000,000,000
Premium: KRW 9,850,000,000 (98.5% of Notional Amount)
Reference Spot Price i: Last prices of "Underlying i" at "T0"
Strike Price: Ki1, Ki2 = 90%, Ki3, Ki4 = 85%, and Ki5, Ki6 = 80% of Reference Spot Price i
Valuation Date (Ti): T1 = T0 + 6 months, T2 = T0 + 12 months, T3 = T0 + 18 months, T4 = T0 + 24 months, T5 = T0 + 30 months, T6 = T0 + 36 months
Fixing Price ij of the Underlying Share i in respect of Tj: The last price of the Underlying i at the Valuation Date Tj
Knock-in Price: Hi = 55% of Reference Spot Price i
Knock-in Event: Applicable. Knock-in Event shall occur when at least one of the price of the Underlying i on the Exchange is below than the Knock-in Price Hi
Option Cash Settlement Amount: On Expiration Date ("T6"), Party A shall pay to Party B an amount in KRW equal to 1) or 2)
1) If all Fixing Prices i6 of Underlying i in respect of T6 are greater than or equal to their correspoding Ki6, an amount of

$$145.00\ \% * \text{Notional Amount}$$

2) If some Fixing Prices of Underlying i in respect of T6 is less than Ki6,
(1) if Knock-in Event has occurred, an amount of

$$\min\left[\frac{\text{Fixing Price i6}}{\text{Reference Spot Price i}}\right] * \text{Notional Amount},$$

(2) if Knock-in Event has NOT occurred, an amount of

$$145.00\ \% * \text{Notional Amount}$$

Early Termination Event: If for ALL Fixing Prices ij (j=1, 2, 3, 4, 5) are equal to or greater than T-th Strike Price Kij, then Party A shall pay to Party B the Early Termination Amount on Tj

Early Termination Amount: Upon occurrence of an Early Termination Event in respect of Valuation Date j, the following amount in KRW shall be the Early Termination Amount

[100.00% + (j * 7.50%)] * Notional Amount

It seems to be little bit complicated. However, it can be simplified as an option on option with digital payoffs. For example, on the first redeemable date, the payoff would be seperated as fixed amount of money(digital, trigger) and not triggered self similar "Two stock step down trigger" by a strike.

Two widely used methods to price it are finite different methods(FDM) and Monte Carlo methods(MC). Not only for these structures but also for almost all exotic pricings, FDM and MC are always important candidates to be used. Pricing using FDM is more powerful for american style products while MC for path-dependent ones. True or false, we believe that using MC is more easy to construct while FDM more accurate. Someone believes that a quant majoring in Mathematics prefers MC while in Physics FDM. However, most of Korea finanacial firm are using FDM for pricing "Two stock step down trigger" since they believe that it is more efficient for getting hedge greeks as well. Consult [2], [4], [5] for more details about FDM and MC.

Monte Carlo methods for general derivatives pricing is based on the fact that geometric Brownian motion,

$$\frac{dS}{S} = (r-q)dt + \sigma dW$$

has the exact solution

$$S_T = S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)T + z\sigma\sqrt{T}},$$

where $z \sim \mathcal{N}(0,1)$. Therefore, we can generate sample paths as much as we want by means of generating a normal random z. Moreover, some fancy tricks such as Cholesky decompositions enable us to get ρ correlated sample paths. So just taking an average of payoffs $V(S_{1,T}, S_{2,T})$ deduced by each sample path and discounting it is all we should do for pricing using MC.

If we consider path-dependent structures, dividing the sample path into several steps is helpful. At the moment considering the dependancy, the sample path memorizes those effects to control the payoff values. For example, when we consider a derivative whose payoff is depend on everyday's close price, steps of sample path should be same as a day. It means that we should generate 365 normal random number if the maturity of that product is 1 year. And lots of structure's depancies are continuous monitored, which means that the step should be infinitely small. Most of quants approximate continuous monitored structure, for instance a barrier option as once-a-day monitored one. Anyway, it is not efficient and accurate.

Fortunately, we know the density functions on maturity for a barrier payoff or a lookback payoff, which enable us not to divide steps. Just taking an average with this density on maturity, that is integrating with this density function drive the price of those structures. The density function for a barrier payoff is derived by a tricky methods called an image solution. Refer Buchen's paper[1].

3. Probability density function for extreme values

The probability density function for extreme values of two correlated Brownian motions is given by He, Keirstead and Rebholz[3]. Suppose that

$$X_i(t) \equiv \alpha_i t + \sigma_i \omega_i(t), \quad t \geq 0,$$

where α_i and σ are constants, ω_i is a Brownian motion and $cov(\omega_1(t), \omega_2(t)) = \rho t$ with constant ρ and

$$\begin{aligned} \underline{X}_i(t) &= \min_{0 \leq s \leq t} X_i(s), \\ \overline{X}_i(t) &= \max_{0 \leq s \leq t} X_i(s). \end{aligned}$$

Then, the following formula holds.

Formula 1. *Define*

$$\begin{aligned} &\mathcal{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, \underline{X}_1(t) \geq m_1, \underline{X}_2(t) \geq m_2) \\ &\qquad \equiv p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) dx_1 dx_2. \end{aligned}$$

Then for $x_1 \geq m_1$, $x_2 \geq m_2$, *where* $m_1 \leq 0$, $m_2 \leq 0$,

$$\begin{aligned} p(x_1, x_2, t \quad &; \quad m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) \\ &= \quad \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} h(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho), \end{aligned}$$

where

$$\begin{aligned} h(x_1, x_2, t \quad &; \quad m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) \\ &= \quad \frac{2}{\beta t} \sum_{n=1}^{\infty} e^{-(r^2 + r_0^2)/2t} \sin \frac{n\pi\theta_0}{\beta} \sin \frac{n\pi\theta}{\beta} I_{(n\pi/\beta)}(\frac{rr_0}{t}) \end{aligned}$$

and

$$\begin{aligned} &a_1 = \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \quad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}, \\ &b = -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2, \\ &\tan\beta = -\frac{\sqrt{1-\rho^2}}{\rho}, \quad \beta \in [0, \pi], \\ &z_1 = \frac{1}{\sqrt{1-\rho^2}} \left[\left(\frac{x_1 - m_1}{\sigma_1} \right) - \rho \left(\frac{x_2 - m_2}{\sigma_2} \right) \right], \quad z_2 = \frac{x_2 - m_2}{\sigma_2}, \\ &z_{10} = \frac{1}{\sqrt{1-\rho^2}} \left[-\frac{m_1}{\sigma_1} + \frac{\rho m_2}{\sigma_2} \right], \quad z_{20} = -\frac{m_2}{\sigma_2}, \\ &r = \sqrt{z_1^2 + z_2^2}, \quad \tan\theta = \frac{z_2}{z_1}, \quad \theta \in [0, \beta], \\ &r_0 = \sqrt{z_{10}^2 + z_{20}^2}, \quad \tan\theta_0 = \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0, \beta]. \end{aligned}$$

Authors give a proof via an appendix. The proof's idea is similar to a single variable's one, called an image solution. With this density, we don't need to divide the sample path steps into a day for pricing "Two stock step down trigger". Digital parts of it can be priced using general MC since they are just options on option on option on $\cdots$ on digital. The remaining part can be dealt as an option on option on option on $\cdots$ on barrier, which can be priced using MC with above densities.

References

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E-mail address: bkseoz@ibk.co.kr