

## A SURVEY ON THE OPTIMAL EXERCISE BOUNDARY OF AMERICAN OPTIONS

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**ABSTRACT.** Unlike European options, American options can be exercised at any time before the expiration date. This fact makes it difficult to analyze the price and the optimal exercise boundary of an American option. This article surveys the literature on the analysis and numerical computations of the optimal exercise boundary for American options. .

### 1. TRADITIONAL METHODS FOR AMERICAN OPTIONS

One of the exciting developments in financial markets over the last 30 years has been the growth of derivatives markets. This explosive growth in the use of derivatives has needed their efficient and accurate valuation. The problem of valuing American options has a large literature in financial engineering and financial mathematics field. Since Black and Scholes (1973) and Merton (1973) derived a closed-form formula for European options, many researchers have been tried to obtain analytical results for American options. Unlike European options, American options can be exercised at any time before the expiration date. This fact makes it difficult to analyze the price and the optimal exercise boundary of American option. The need of determining the optimal exercise boundary as part of the solution has made the pricing of American options much harder than pricing their European counterparts. So far a closed-form formula for the price of American options and the optimal exercise boundary has not been found except for some special cases. For this reason, many efforts have been concentrated on developing analytic approximation methods and numerical methods for the American option value and the optimal exercise boundary.

Quadratic approximation method of MacMillan (1986) and Barone-Adesi and Whaley (1987), compound option approach of Geske and Johnson (1984) and Bunch and Johnson (1992), LUBA (lower and upper bound approximation) of Broadie and Detemple (1996) are examples of traditional analytic approximation methods.

There are three kinds of traditional numerical methods: binomial method, finite difference method and Monte Carlo simulation method. The binomial method of Cox, Ross, and Rubinstein (1979) and the finite difference method of Brennan and Schwartz (1977, 1978) are the earliest numerical methods and are still widely

used. Monte Carlo simulation method have been applied to the valuation of American options by Tilley (1993), Barraquand and Martineau (1995), Grant, Vora, and Weeks (1996), Broadie and Glasserman (1997), Broadie, Glasserman, and Jain (1997), Longstaff and Schwartz (2001), Moreno and Navas (2003), Ibanez and Zapatero (2004) and Stentoft (2004) among others. Longstaff and Schwartz (2001) use least squares to estimate the conditional expected payoff to the option holder from continuation.

## 2. INTEGRAL REPRESENTATION

Consider an American put option with exercise price of  $K$  that expires at time  $T$ . The price of the underlying asset  $S(t)$  follows a lognormal diffusion process.  $r$  is the rate of interest,  $\sigma$  is the volatility,  $\delta$  is the dividend yield,  $\tau = T - t$  is the time to maturity. Denote the American put option price by  $P(S, \tau)$  and the optimal exercise boundary by  $B_\tau^*$  as a function of the time to maturity.

Kim (1990), Jacka (1991) and Carr, Jarrow, and Myneni (1992) have obtained the following "integral representation" formula for an American put:

$$(1) \quad P(S, \tau) = p(S, \tau) + \int_0^\tau \left[ rKe^{-r(\tau-s)} N\left(-d_2(S, B_s^*, \tau-s)\right) - \delta Se^{-\delta(\tau-s)} N\left(-d_1(S, B_s^*, \tau-s)\right) \right] ds$$

where

$$d_1(x, y, \tau-s) = \frac{\log(x/y) + (r - \delta + \sigma^2/2)(\tau-s)}{\sigma\sqrt{\tau-s}}$$

$$d_2(x, y, \tau-s) = d_1(x, y, \tau-s) - \sigma\sqrt{\tau-s}$$

Here  $N(\bullet)$  is the standard cumulative normal distribution function and  $p(S, \tau)$  is the price of the corresponding Black and Scholes (1973) European put option formula. In this integral representation formula, the American option price is the sum of otherwise identical European option price and the early exercise premium that is expressed as an integral.

The integral representation formula (1) for the American option requires the determination of the optimal exercise boundary for its implementation. To determine the optimal exercise boundary, we can apply boundary conditions to generate nonlinear integral equations and obtain:

$$(2) \quad K - B_\tau^* = p(B_\tau^*, \tau) + \int_0^\tau \left[ rKe^{-r(\tau-s)} N\left(-d_2(B_\tau^*, B_s^*, \tau-s)\right) - \delta B_\tau^* e^{-\delta(\tau-s)} N\left(-d_1(B_\tau^*, B_s^*, \tau-s)\right) \right] ds$$

Equation (2) is a nonlinear Volterra integral equation of the second kind [see Linz (1985)]. Little, Pant, and Hou (2000) derives an alternative integral equation for the optimal exercise boundary of American put options. However, it is not possible to obtain the optimal exercise boundary in explicit form. Kim (1994) develops a simple approximation formula for the optimal exercise boundary of American futures options. Kim and Byun (1994) present the properties of the optimal exercise boundary in a binomial option pricing model and develop an efficient recursive valuation method. Kim, Byun, and Lim (2004) study the numerical properties of the optimal exercise boundary in the case of deterministic volatility function. Byun (2005) investigates the properties of the integral equation (2) and presents

a simple upper bound for the optimal exercise boundary of American put options. Several numerical methods for valuing American options, based on this integral representation formula, are proposed by Broadie and Detemple (1996), Huang, Subrahmanyam, and Yu (1996), Ju (1998), AitSahlia and Lai (2001), and Kallast and Kivinukk (2003) among others.

### 3. INFINITE SERIES SOLUTION

Zhu (2006b) derived an exact solution for the American option value and the optimal exercise boundary in the form of an infinite series expansion using the homotopy analysis method. However, the implementation of Zhu's (2006b) infinite series solution is very complex and time consuming. So, Zhu (2006a, 2011) and Zhu and He (2007) proposed analytical approximation methods for the value of American options and the optimal exercise boundary utilizing the infinite series solution. Zhu (2011) uses Zhu's (2006a) approximation formula as an initial guess for the optimal exercise boundary, an improved boundary is then achieved by setting a null value of the theta of option on the optimal exercise boundary.

### 4. OPTIMAL EXERCISE BOUNDARY AT NEAR EXPIRY

Asymptotic expansions of the optimal exercise boundary near the expiration time has been investigated extensively in Kuske and Keller (1998), Stamicar et al. (1999), Bunch and Johnson (2000), Knessl (2001), Evans, Kuske, and Keller (2002), Chen and Chadam (2007), Chen et al (2008) among others. Kuske and Keller (1998), Stamicar et al. (1999) and Bunch and Johnson (2000) derive very similar asymptotic expansions of the optimal exercise boundary. Chen and Chadam (2007) provide four approximations for the optimal exercise boundary: one is explicit and is valid near expiry; two others are implicit involving inverse functions; the fourth is an ODE initial value problem. In a subsequent study Chen et al. (2008) show that the optimal exercise boundary for the American put option with non-dividend-paying asset is convex. With this convexity result, they provide an accurate asymptotic behavior for the optimal exercise boundary near expiry. Lauko and Sevcovic (2010) compares various analytical and numerical approximation methods for calculating the optimal exercise boundary of the American put option paying zero dividends.

The continuity and differentiability of the optimal exercise boundary of the American put option in jump diffusion models have been studied by Pham (1997), Yang et al. (2006), Lamberton and Mikou (2008) and Bayraktar and Xing (2009).

### 5. AMERICAN STYLE PATH DEPENDENT OPTIONS

From practical point of view, the financial derivatives markets have expanded not only in terms of size, but also in terms of variety. In order to meet the diversified needs of consumers, financial derivatives become more complicated and the closed-form pricing solutions are not available for them. Especially, path dependent derivatives with American style exercise features have the complicated optimal exercise region. Path dependent derivatives is a class of financial derivatives whose payoff diagram depends not only on the underlying asset price but also on the path of underlying asset prices over some predetermined time interval. Therefore, it is needed to develop simple, efficient, and easy to implement approximation and numerical methods for the optimal exercise region of American-style path dependent derivatives.

In this line of research, Sevcovic (2008), Sevcovic and Takac (2011) and Bokes and Sevcovic (2011) discuss an application of the transformation method for the pricing equation for American type of Asian options. Sevcovic and Takac (2011) investigate the dependence of the optimal exercise boundary on model parameters for various types of averaging methods. Dai and Kwok (2006) analyze the monotonicity properties of the optimal stopping regions of American path-dependent options with either the Asian feature or lookback feature. From the monotonicity properties, Dai and Kwok (2006) show that the exercise region of an American lookback option is always contained in the exercise region of its Asian option counterpart. Kim et al (2003) develop a lattice method to value arithmetic average reset options and the method can be applied to American style derivatives with an arithmetic average reset feature. A recent paper of Bokes (2010, 2011) analyzes the asymptotic behavior of the optimal exercise boundary at near expiry for American-style Asian and lookback options.

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