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HEAT KERNEL METHOD AND FOURIER ANALYSIS ON HYPERFUNCTIONS

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Abstract. The Sato theory of hyperfunctions and Fourier hyperfunctions is a really natural extension of the Schwartz theory of distributions and tempered distributions. We show that the naturalness of hyperfunctions comparing our results in hyperfunctions and the corresponding results in distributions in such areas as the characterization of test function spaces for Fourier transformation, both periodic and almost periodic distributions and hyperfunctions and Bochner-Schwartz theorem for (conditionally) positive definite distributions and hyperfunctions. We have one important guiding theme to compare distributions and hyperfunctions. Estimates related to distributions are *tempered* or *polynomially increasing* and estimates related to hyperfunctions are of *infra-exponential growth*. Also, in order to define periodicity, (conditionally) positive definiteness for the hyperfunctions we make use of Matsuzawa's heat kernel method which represents various generalized functions including distributions and hyperfunctions as the limits of initial values of the solutions of the heat equation satisfying suitable growth conditions.

1. Introduction

L. Schwartz introduced the theory of distributions

(1) to *differentiate freely* continuous (or locally integrable) functions and
(2) to *take Fourier transform* of tempered (or polynomially increasing) functions or distributions.

For (1) he introduced the space C_c^∞ (or $\mathcal{D}$) of infinitely differentiable functions with compact support, for (2) the Schwartz space $\mathcal{S}$ of *rapidly decreasing* functions.

We define the space $\mathcal{D}'(X)$ of distributions as the space of continuous of linear forms on $C_c^\infty(X)$ with the inductive limit topology. More precisely. $u : C_c^\infty(X) \to \mathbb{C}$, linear, is a distribution if for every compact set $K \subset X$ there exist constants $C,\ k$ satisfying

$$|u(\phi)| \le C \sum_{|\alpha|\le k} \sup |\partial^\alpha \phi|, \quad \phi \in C_c^\infty(K).$$

Here we use the multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

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If $C_c^\infty(X)$ is replaced by a dense subspace with a stronger topology then we obtain a larger space of generalized functions. For example, we define the space $\mathcal{C}'_M(X)$ of ultradistributions as the space of continuous of linear forms on the non quasianalytic Denjoy–Carleman class $\mathcal{C}_M(X)$. The situation is quite different in the quasianalytic case since there does not exist any element with compact support. In this case the dual space of $\mathcal{C}_M(X)$ can be regarded as the space of elements with compact support in a distribution theory as explained in [H]. When $\mathcal{C}_M(X)$ is the real analytic class the largest space is obtained and this space of hyperfunctions was introduced by Sato.

As the Denjoy-Carleman class lies between the C^∞ class and the class of analytic functions, non-quasianalytic Denjoy-Carleman classes and quasianalytic classes have similar properties as the C^∞ class and the analytic class, respectively by the Denjoy-Carleman theorem. Thus, a space of non-quasianalytic ultradistributions (usually, called ultradistributions), which is a dual space of non-quasianalytic class behaves very much alike the space of Schwartz distributions(see [Ko] for this).

Also, a space of quasianalytic ultradistributions (sometimes, called infra-hyperfunctions), which is a dual space of a quasianalytic class, has been supposed to behave very much alike the space of hyperfunctions.

For Fourier transformation Sato and Kawai [Ka] also introduced the space $\mathcal{P}_*$ of complex analytic and exponentially decreasing functions in a tubular neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ to take Fourier transform of infra-exponentially (or slowly) increasing functions or hyperfunctions. They denote by $\mathcal{Q}$ the strong dual space of $\mathcal{P}_*$ and name its elements *Fourier hyperfunctions.* Also, K.H. Kim, S.Y. Chung and D. Kim [KCK] introduced a real version $\mathcal{F}$ which they named the Sato space, of the space $\mathcal{P}_*$ and showed that they are isomorphic. Thus the global theory of the Fourier hyperfunctions is nothing but the duality theory for the Sato space $\mathcal{F}$.

The theory of hyperfunctions and Fourier hyperfunctions is a really natural extension of the Schwartz theory of distributions and tempered distributions. But, due to the sheaf theoretical definition of hyperfunctions, the Sato theory has scared many analysts who are not familiar with the sheaf cohomology away from the topic.

In this article we will show the naturalness of hyperfunctions comparing our results in the theory of hyperfunctions and the corresponding results in the Schwartz theory of distributions in such areas as

(1) the characterization of test function spaces in terms of Fourier transformations
(2) generalized functions as initial values of solutions of the heat equation
(3) the growth of Fourier coefficients of periodic hyperfunctions
(4) almost periodic hyperfunctions
(5) Positive definiteness and Bochner-Schwartz theorem.

To obtain the above theorems of global nature in hyperfunctions we make use of the heat kernel method of Matsuzawa effectively which represents various generalized functions as initial values of smooth solutions of the heat equation satisfying suitable growth condition.

We have only one very important guiding theme to compare distributions and hyperfunctions, and ultradistributions both quasianalytic and non quasianalytic which lie between distributions and hyperfunctions.

In the Schwartz theory of distributions

(1) estimates related to the space of test functions are *rapidly decreasing* and

(2) estimates related to the dual space of distributions are *tempered* or *polynomially increasing.*

On the other hand, in the Sato theory of hyperfunctions

(1) estimates related to the space of test functions are *exponentially decreasing* and

(2) estimates related to the dual space of hyperfunctions are *slowly increasing* or *infra-exponentially increasing.*

Our guiding theorem is also satisfied in the following case of the Paley–Wiener–Schwartz theorem for distributions and the Paley–Wiener–Ehrenpreis theorem for hyperfunctions.

Theorem 1.1 (Paley-Wiener-Schwartz [H1])**.** *Let $K \subset \mathbb{R}^n$ be a compact convex subset with supporting function H and let $u \in \mathcal{E}'^{(N)}(K)$. Then*

$$|\hat{u}(\zeta)| \leq C(1+|\zeta|)^N e^{H(Im\,\zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.1}$$

Conversely, every entire analytic function in $\mathbb{C}^n$ satisfying an estimate of the form (1.1) is the Fourier-Laplace transform of a distribution with support contained in K.

If $u \in C_c^\infty(K)$ then for all $N \in \mathbb{N}$, there exists C such that every N a constant C_N such that

$$|\hat{u}(\zeta)| \leq C_N(1+|\zeta|)^{-N} e^{H(Im\,\zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.2}$$

Conversely, every entire analytic function in $\mathbb{C}^n$ satisfying (1.2) is the Fourier-Laplace transform of a function in $C_c^\infty(K)$.

Theorem 1.2 (Paley-Wiener-Ehrenpreis [H1])**.** *Let $K \subset \mathbb{R}^n$ be a compact convex subset with supporting function H and let u be an analytic functional with support in K. Then the Fourier-Laplace transform*

$$\hat{u}(\zeta) = u(\exp(-i\langle\cdot,\zeta\rangle)), \quad \zeta \in \mathbb{C}^n,$$

is an analytic function such that for every $\epsilon > 0$ there exists C_ϵ such that

$$|\hat{u}(\zeta)| \leq C_\epsilon e^{\epsilon|\zeta|} e^{H(Im\,\zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.3}$$

Conversely, every entire analytic function in $\mathbb{C}^n$ satisfying an estimate of the form (1.3) is the Fourier-Laplace transform of a unique $u \in \mathcal{A}'(K)$.

The Roumieu–Beurling theory of ultradistributions fills the gap between distributions and hyperfunctions.

2. Characterization of various test functions spaces via Fourier transform

In this section we give more symmetric and better characterization via Fourier transform, of the Schwartz space of rapidly decreasing functions, the Sato space of real analytic and exponentially decreasing functions, the Beurling–Björck space of test functions for tempered ultradistributions and the Gel'fand–Shilov spaces of type S, type W and generalized type S as in [S]. L. Schwartz introduced the Schwartz space $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ of all infinitely differentiable functions φ satisfying

$$\sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty. \tag{2.1}$$

for all multi-indices $\alpha,\ \beta$ in his famous treatise [S].

If $\varphi \in S$ then it follows from (2.1) that

$$\sup_x |x^\alpha \varphi(x)| < \infty, \quad \sup_x |\partial^\beta \varphi(x)| < \infty$$

for all multi-indices $\alpha,\ \beta$. In [CCK1] we show that the converse also holds true. In other words, the condition (2.1) characterizes the Schwartz space $\mathcal{S}$ as follows:

Theorem 2.1 ([CCK1]). *For the Schwartz space $\mathcal{S}$ the following statements are equivalent.*

(1) $\varphi \in \mathcal{S}$.
(2) $\sup_x |x^\alpha \varphi(x)| < \infty, \quad \sup_x |\partial^\beta \varphi(x)| < \infty$
for every α and β.
(3) $\sup_x |x^\alpha \varphi(x)| < \infty, \quad \sup_\xi |\xi^\beta \hat{\varphi}(\xi)| < \infty,$
for every α and β.

Here, $\hat{\varphi}$ is the Fourier transform of φ, i.e.,

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx.$$

The proof of Theorem 2.1 in [CCK1] is based on the combination of induction and the method of the proof of the Sobolev embedding theorem. We also give a very elementary proof of the Theorem 2.1 in [CKLs] based on the elementary inequality of Landau concerning the estimates of derivatives.

To obtain the similar results for Fourier hyperfunctions we recall briefly the original complex version of Sato–Kawai and our real definition of the Sato space.

Definition 2.2 (Sato–Kawai, [Ka]). A complex valued function $\varphi(z)$ is in $\mathcal{P}_*$ if $\varphi(z)$ is holomorphic and exponentially decreasing in a tubular neighborhood $\mathbb{R}^n + i\{|y| \leq r\}$, for some r, of $\mathbb{R}^n$, i.e., for some $k > 0$

$$\sup_{z \in \mathbb{R}^n + i\{|y| \leq r\}} |\varphi(z)| \exp k|z| < \infty.$$

Definition 2.3 ([CCK2]). A real valued function φ is in $\mathcal{F}$ if $\varphi \in C^\infty(\mathbb{R}^n)$ and if there exist $h,\ k > 0$ satisfying

$$|\varphi|_{k,h} = \sup_{\alpha,x} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|}\alpha!} \exp k|x| < \infty.$$

The motivation to introduce Definition 2.3 is the following Pringsheim's characterization of real analytic functions:

$C^\infty(X)$ is analytic if and only if for every compact subset $K \subset\subset X$ there exists positive constants $C,\ r = 1/h$ satisfying

$$\sup_{K,\alpha} \frac{|\partial^\alpha f(x)| r^\alpha}{\alpha!} = \sup_{K,\alpha} \frac{|\partial^\alpha f(x)|}{\alpha! h^{|\alpha|}} < \infty.$$

We can easily show that the Sato space $\mathcal{F}$ is isomorphic to the space $\mathcal{P}_*$ introduced by Sato and Kawai and denote by $\mathcal{F}'$ the strong dual space of $\mathcal{F}$ and call its elements *Fourier hyperfunctions.*

We are now in a position to state a characterization theorem for the Sato space in terms of Fourier transform.

Theorem 2.4 ([CCK2]). *For the Sato space $\mathcal{F}$ the following statements are equivalent.*

(1) *$\varphi \in \mathcal{F}$.*
(2) *There exist constants $h,\ k > 0$ such that*

$$\sup_x |\varphi(x)| \exp k|x| < \infty, \quad \sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi| < \infty.$$

Remark 2.5. We can easily see that our guiding theme is satisfied in this case comparing the spaces $\mathcal{S}$ and $\mathcal{F}$ which are both invariant under the Fourier transformation as follows:

(i) The space $\mathcal{S}$ consists of all locally integrable function φ such that φ itself and its Fourier transform $\hat{\varphi}$ are both rapidly decreasing.
(ii) The space $\mathcal{F}$ consists of all locally integrable function φ such that φ itself and its Fourier transform $\hat{\varphi}$ are both exponentially decreasing.

Also, we can give characterizations e the Gel'fand–Shilov spaces of (generalized) type S and type W in a more symmetric way by means of the Fourier transformation, which are generalizations of Theorem 2.1 and Theorem 2.4.

We first introduce the *Gel'fand–Shilov spaces of generalized type S* and *type S*. Let $M_p,\ p = 0, 1, 2, \ldots,$ be a sequence of positive numbers satisfying conditions (M.1) and (M.2) in [Ko].

Definition 2.6 ([GS2])**.** The Gel'fand-Shilov spaces $S_{M_p}^{N_p}$ consist of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ satisfying the following estimates, respectively,

$$\sup_x |x^\alpha \partial^\beta \varphi(x)| \le C A^{|\alpha|} B^{|\beta|} M_{|\alpha|} N_{|\beta|}$$

for some positive constants A B for all multi-indices α and β.

In particular, if $M_p = p!^r$ and $N_p = p!^s$ then we denote the spaces $S_{M_p}^{N_p}$ by S_r^s and call these spaces the *Gel'fand–Shilov spaces of type S*.

Let $M(x)$ and $\Omega(y)$ be differentiable functions on $[0, \infty)$ satisfying the condition (K): $M(0) = \Omega(0) = 0$ and their derivatives are continuous, increasing and tending to infinity.

We now define the *Gel'fand–Shilov spaces of type W* as in [GS2].

Definition 2.7 ([GS2])**.** The space W_M^Ω consists of all entire analytic functions $\varphi(\zeta)$ on $\mathbb{C}^n$ satisfying the estimate $|\varphi(\xi + i\eta)| \le C \exp(-M(a|\xi|) + \Omega(b|\eta|))$ for some $a,\ b > 0$.

For a given sequence M_p we define its *associated* function $M(t)$ by $M(t) = sup_p \log t^p M_0/M_p$. When $M : [0, \infty) \to [0, \infty)$ is a convex and increasing function we define its *Young conjugate* or *Legendre transform* M^* by $M^*(\rho) = sup_x(x\rho - M(x))$.

We are now in a position to state the more symmetric characterization theorems for the above Gel'fand-Shilov spaces in terms of Fourier transform.

Theorem 2.8 ([CCK4])**.** (1) *For the Gel'fand-Shilov space S_α^β of type S the following statements are equivalent.*

(i) *$\varphi \in S_\alpha^\beta$.*
(ii) *$\sup_x |\varphi(x)| \exp k|x|^{1/\alpha} < \infty,\ \sup_\xi |\hat{\varphi}(\xi)| \exp h|\xi|^{1/\beta} < \infty$ for some $h,\ k > 0$.*

(2) *For the space $S_{M_p}^{N_p}$ the following statements are equivalent.*

(i) *$\varphi \in S_{M_p}^{N_p}$;*

(ii) $\sup_x |\varphi(x)| \exp M(ax) < \infty, \quad \sup_\xi |\hat{\varphi}(\xi)| \exp N(b\xi) < \infty$
for some $a,\ b > 0$, *where* $M(x)$ *and* $N(\xi)$ *are associated functions of* M_p *and* N_p, *respectively.*

(3) *For the space* W_M^Ω *the following statements are equivalent.*
(i) $\varphi \in W_M^\Omega$;
(ii) $\sup_x |\varphi(x)| \exp M(a|x|) < \infty, \quad \sup_\xi |\hat{\varphi}(\xi)| \exp \Omega^*(b|\xi|) < \infty$
for some $a,\ b > 0$, *where* Ω^* *is the Young conjugate of* Ω.

G. Björck introduced the Beurling–Björck space $\mathcal{S}_\omega$ of test functions to extend the space of tempered distributions in [Bj] as follows.

Definition 2.9 ([B]). By $\mathcal{S}_\omega$ or $\mathcal{S}_\omega(\mathbb{R}^n)$ we denote the set of all $\varphi \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that

$$\begin{aligned} p_{\alpha,\lambda}(\varphi) &= \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} |\partial^\alpha \varphi(x)| < \infty, \\ \pi_{\alpha,\lambda}(\varphi) &= \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)} |\partial^\alpha \hat{\varphi}(\xi)| < \infty \end{aligned}$$

for each multi-index α and each non-negative number λ, where the continuous weight function ω on $\mathbb{R}^n$ satisfies the following:

(γ) $\omega(\xi) \geq a + b\log(1+|\xi|), \quad \xi \in \mathbb{R}^n$ for some real a and positive b,
(δ) $\omega(\xi) = \Omega(|\xi|)$ with Ω concave on $[0,\infty)$.

We finally give symmetric characterization of the above space which shows that the condition on derivatives in the original definition is redundant.

Theorem 2.10 ([CKLs]). *Let* ω *be a function satisfying the conditions* (γ) *and* (δ) *and let* φ *and* $\hat{\varphi}$ *be* C^∞ *functions. Then the following are equivalent.*

(1) $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} |\partial^\alpha \varphi(x)| < \infty, \quad \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)} |\partial^\alpha \hat{\varphi}(\xi)| < \infty$
(2) $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} |\varphi(x)| < \infty, \quad \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)} |\hat{\varphi}(\xi)| < \infty$

for all $\lambda > 0$ *and* α.

3. Generalized functions as initial values of solutions of the heat equation

We characterize the various generalized functions including distributions, tempered distributions, hyperfunctions and Fourier hyperfunctions as the initial values of the solutions of the heat equation.

We first introduce analytic functionals to define hyperfunctions following A. Martineau.

Definition 3.1. If $K \subset \mathbb{C}^n$ is a compact set, then $\mathcal{A}'(K)$, the space of analytic functionals carried by K, is the space of linear forms u on the space of A of entire analytic functions in $\mathbb{C}^n$ such that for every neighborhood ω of K

$$|u(\varphi)| \leq C_\omega \sup_\omega |\varphi|, \quad \varphi \in \mathcal{A}.$$

$$|u(\varphi)| \leq C \sup_{\alpha}{}^{\in K} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|}\alpha!}, \quad \varphi \in \mathcal{A}.$$

We now define the space $\mathcal{B}$ of hyperfunctions.

Definition 3.2. Let Ω be a bounded open set in $\mathbb{R}^n$. Then the space $\mathcal{B}(\Omega)$ of hyperfunctions is defined by $\mathcal{B}(\Omega) = \mathcal{A}'(\bar{\Omega})/\mathcal{A}'(\partial\Omega)$.

Theorem 3.3. *Let Ω_j, $j = 1, 2, \ldots,$ be bounded open subsets of $\mathbb{R}^n$ such that $\Omega = \cup_{i=1}^{\infty} \Omega_j$. If $u_j \in \mathcal{B}(\Omega_j)$ and for all i, j we have $u_i = u_j$ in $\Omega_i \cap \Omega_j$ (that is, supp $(u_i - u_j) \cap \Omega_i \cap \Omega_j = \emptyset$) then there is a unique $u \in \mathcal{B}(\Omega)$ such that the restriction of u to Ω_j is equal to u_j.*

It follows from the above theorem that a hyperfunction $u \in B(\mathbb{R}^n)$ can be regarded as a collection of $u_j \in \mathcal{A}'(\overline{\Omega_j})$ with $\mathbb{R}^n = \cup_j \Omega_j$, and we write $u = (u_j)$.

Hörmander [H1] represents an analytic functional $u \in \mathcal{A}'(K)$ as the initial value of harmonic functions in $\mathbb{R}^{n+1} \backslash (K \times 0)$ to introduce hyperfunctions in [H1] as follows: Let $\Delta E = \delta$ in $\mathbb{R}^{n+1}$, where $X = (x, t)$, i.e.,

$$K(X) = -\frac{1}{(n-1)|S^{n+1}||X|^{n+1}}.$$

Also, let

$$P(X) = \partial E/\partial t = \frac{1}{|S^{n+1}|}\frac{t}{|X|^{n+1}}.$$

For $u \in C_c^0(K)$ or $\mathcal{A}'(K)$ we correspond $U = P * (u \otimes \delta)$ which is a kind of defining function and odd in t and harmonic outside $K \times 0$.

Furthermore, $U \to \pm u/2$ as $t \to \pm 0$.

Conversely, if $K \subset \mathbb{R}^n$ and U is harmonic in $\mathbb{R}^{n+1}/(K \times 0)$ and odd in t then there exists a unique $u \in \mathcal{A}'(K)$ which can be regarded as an initial value of U.

On the other hand, T. Matsuzawa improved Hörmander's method by using the following heat kernel instead of the Poisson kernel in ([M1, M2]).

Let $E(x, t)$ be the n-dimensional heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Note that $E(\cdot, t)$ belongs to $\mathcal{F}$ for each $t > 0$.

Then $U(x, t) = u_y(E(x - y, t))$, $x \in \mathbb{R}^n$, $t > 0$ is well defined for $u \in \mathcal{A}'(K)$ or $u \in \mathcal{F}'$ and is called a *Gauss transform* or a *defining* function of u.

We now represent distributions, hyperfunctions as in Matsuzawa [M1] and Fourier hyperfunctions as in Kim–Chung–Kim [KCK], as the initial values of smooth solutions of the heat equation.

Theorem 3.4 ([M1]). (i) Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then there is a defining function $U(x,t) \in C^\infty(\mathbb{R}_+^{n+1})$ satisfying the following conditions:

$$(\partial/\partial t - \Delta)U(x, t) = 0 \quad \text{in} \quad \mathbb{R}_+^{n+1}. \tag{3.1}$$

For any compact set $K \subset \mathbb{R}^n$ there exist positive constants $N = N(K)$ and C_K such that

$$|U(x, t)| \le C_K\, t^{-N}, \quad t > 0,\ x \in K \tag{3.2}$$

and $U(\,\cdot\,, t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in C_c^\infty$

$$u(\varphi) = \lim_{t \to 0^+} \int U(x, t)\varphi(x)\, dx. \tag{3.3}$$

Conversely, let $U(x, t) \in C^\infty(\mathbb{R}_+^{n+1})$ satisfy (3.1) and (3.2). Then there exists a unique $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfying (3.3).

(ii) Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then there is a defining function $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ satisfying the heat equation (3.1) and the following conditions: For every compact subset $K \subset \mathbb{R}^n$ and for every $\epsilon > 0$ there exists a constant $C_{\epsilon,K} > 0$ such that

$$|U(x,t)| \le C_{\epsilon,K} \exp(\epsilon/t), \quad t > 0,\, x \in K, \tag{3.4}$$

and

$$U(\,\cdot\, t) \to u \quad \text{as } t \to 0^+ \tag{3.5}$$

in the sense that $U(x,t) - U_j(x,t) \Longrightarrow 0$ as $t \to 0^+$ in $\Omega_j,\ j = 1,2,\ldots$, where $\Longrightarrow$ means weak uniform convergence in Ω_j and U_j is the defining function of $u_j \in \mathcal{A}'(\bar{\Omega}_j)$ and $u = (u_j) \in \mathcal{B}(\mathbb{R}^n)$ with $\mathbb{R}^n = \cup_j \Omega_j$.

Conversely, let $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ satisfy (3.1) and (3.4). Then there exists a unique $u \in \mathcal{B}(\mathbb{R}^n)$ satisfying (3.5).

Remark 3.5. (i) [M2] If we replace the growth condition (3.4) by

$$|U(x,t)| \le C(1+|x|)^M 1/t^N$$

then we obtain the case for Fourier hyperfunctions.

(ii) [KCK] If we replace the growth condition (3.4) by

$$|U(x,t)| \le C_\epsilon \exp\left[\epsilon(|x| + 1/t)\right]$$

then we obtain the case for Fourier hyperfunctions.

4. Periodic hyperfunctions and distributions

It is well known in the theory of distributions that In this paper we show that the following results:

1. (1) Every periodic hyperfunction is a bounded hyperfunction.
2. (2) Every periodic hyperfunction can be represented as an infinite sum of derivatives of bounded continuous periodic functions.
3. (3) Fourier coefficients c_α of periodic hyperfunctions are tempered or polynomially increasing in $\mathbb{R}^n$, i.e., there exists a positive integer N satisfying $|c_\alpha| < C|\alpha|^N$.

In this paper we show that the following results:

1. (1) Every periodic hyperfunction is a bounded hyperfunction.
2. (2) Every periodic hyperfunction can be represented as an infinite sum of derivatives of bounded continuous periodic functions.
3. (3) Fourier coefficients c_α of periodic hyperfunctions are of infra-exponential growth in $\mathbb{R}^n$, i.e., $|c_\alpha| < Ce^{\epsilon|\alpha|}$ for every $\epsilon > 0$. Our result contains the result of Sato and Helgason which deals with the case of $\mathbb{R}^1$ (see [Sa], [He]).

In the proof of these results we use the heat kernel method and introduce the space $\mathcal{B}_{L^p}$ of *Sato hyperfunctions of L^p growth* which generalizes the space $\mathcal{D}'_{L^p}$ of Schwartz distributions of L^p growth. This *heat kernel method*, which represent the above generalized functions as the initial values of smooth solutions of heat equations as in Matsuzawa [M1], Kim-Chung-Kim [KCK] and Chung-Kim [CsK1], can overcome difficulties due to the sheaf theoretical definition of the hyperfunctions when we are dealing with the analytical and global properties in the theory of hyperfunctions. Applying this idea we relate the periodic hyperfunctions to the periodicity of its defining function. Note that also in the case of distributions periodicity in terms of the defining function for the distributions and the original definition for periodic distributions coincide naturally.

We first introduce the new space $\mathcal{B}_{L^p}$ of hyperfunctions of L^p growth which is a natural generalization of $\mathcal{D}'_{L^p}$ for hyperfunctions and will be used to characterize the periodic hyperfunctions.

Definition 4.1 ([CKLe]). We denote by $\mathcal{A}_{L^q}$ $(1 \le q < \infty)$ the space of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\|\varphi\|_{L^q,h} = \sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^q}}{h^{|\alpha|}\alpha!} < \infty$$

for some constant $h > 0$. We say that $\varphi_j \to 0$ in $\mathcal{A}_{L^q}$ as $j \to \infty$ if there is a positive constant h such that

$$\sup_\alpha \frac{\|\partial^\alpha \varphi_j\|_{L^q}}{h^{|\alpha|}\alpha!} \to 0 \quad \text{as } j \to \infty.$$

We denote by $\mathcal{B}_{L^p}$ $(1 < p \le \infty)$ the dual of $\mathcal{A}_{L^q}$, where $1/p+1/q = 1$. In particular, every element in $\mathcal{B}_{L^\infty}$ is called a *bounded hyperfunction*. Note that every element in $\mathcal{D}'_{L^\infty}$ is called a *bounded distribution*.

We also represent these hyperfunctions of L^p growth as the limit the initial values of solutions of the heat equation.

Theorem 4.2 ([CKLe]). *Let $u \in \mathcal{B}_{L^p}$. Then the defining function $U(x,t)$ of u belongs to $C^\infty(\mathbb{R}^{n+1}_+)$ and satisfies the following:*

$$(\partial_t - \Delta)U(x,t) = 0 \quad \text{in } \mathbb{R}^{n+1}_+ : \tag{4.1}$$

For every $\epsilon > 0$ there exists a constant $C > 0$ such that

$$\|U(x,t)\|_{L^p(\mathbb{R}^n_x)} \le C\, e^{\epsilon/t} \quad \text{in } \mathbb{R}^{n+1}_+, \tag{4.2}$$

and $U(\,\cdot\,,t) \to u$ as $t \to 0^+$ in the sense that

$$u(\phi) = \lim_{t \to 0^+} \int_{\mathbb{R}^n} U(x,t)\phi(x)dx, \quad \phi \in \mathcal{A}_{L^q}.$$

Conversely, every C^∞ function defined in $\mathbb{R}^{n+1}_+$ satisfying condition (4.1) and (4.2) can be written as

$$U(x,t) = u_y(E(x-y,t)) \quad \text{in } \mathbb{R}^{n+1}_+$$

with a unique element $u \in \mathcal{B}_{L^p}$.

We are now in a position to state a definition of periodic hyperfunctions u in terms of the collection of analytic functionals with compact support $u_j \in \mathcal{A}'(\Omega_j)$ along lines proposed by Martineau. Also, we relate this definition of periodic hyperfunctions and the periodicity of the defining function U of u as in Theorem 4.2.

We now give a natural definition of periodic hyperfunctions.

Definition 4.3 ([CKLe]). A hyperfunction $u = (u_j)_{j \in \mathbb{Z}} \in B(\mathbb{R}^n)$, where $u_j \in \mathcal{A}'(\overline{\Omega_j})$ with $\mathbb{R}^n = \cup_j \Omega_j$ is *periodic* if $\tau_\alpha u = (\tau_\alpha u_j) = u$ for all $\alpha \in \mathbb{Z}^n$. Here, $\tau_\alpha u_j \in \mathcal{A}'(\tau_\alpha \overline{\Omega_j})$ with $\tau_\alpha \Omega_j = \{x + \alpha | x \in \Omega_j\}$.

¿From this definition we can easily obtain the following

Theorem 4.4 ([CKLe]). *If a hyperfunction u is periodic then the defining function $U(x,t)$ of u is also periodic.*

Making use of the periodicity of the defining function U of a periodic hyperfunction u we now prove the following

Theorem 4.5 ([CKLe]). *Every periodic hyperfunction is a bounded hyperfunction.*

Remark 4.6. It follows from Theorem 4.5 that every periodic hyperfunction u is a periodic Fourier hyperfunction, i.e., u is periodic in $\mathcal{F}'$. Therefore, we obtain that $\tau_\alpha u(\varphi) = u(\varphi)$ for all $\alpha \in \mathbb{Z}^n,\ \varphi \in \mathcal{F}$.

We now state that a periodic hyperfunction can be expanded as a Fourier series with coefficients of infra-exponential growth, which generalizes the results of Gorbačuk's [G], [GG] and the similar result for distributions.

Theorem 4.7 ([CKLe]). *The trigonometric series $u = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha e^{2\pi i\alpha\cdot x}$ is the Fourier series of a periodic hyperfunction, i.e., $\tau_\alpha u = u$ for all $\alpha \in \mathbb{Z}^n$ if and only if for every $\epsilon > 0$ there exists a constant $C > 0$ such that $|c_\alpha| \le Ce^{\epsilon|\alpha|}$ for every $\alpha \in \mathbb{Z}^n$, where $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$.*

5. Almost periodic hyperfunctions and distributions

Let $f(x)$ be a complex valued continuous function defined on $\mathbb{R}$. A number τ is called an ϵ-*almost period* of $f(x)$ if $\sup_{-\infty<x<\infty} |\, f(x+\tau) - f(x)\,| \le \epsilon$. If for any $\epsilon > 0$ there exists a number $l(\epsilon)$ such that every intervals of length $l(\epsilon)$ contains an ϵ-almost period of f, then $f(x)$ is said to be *almost periodic*.

It is well known that the following three statements are equivalent:

(1) f is an almost periodic function.
(2) The set of translations f_h for $h \in \mathbb{R}$ forms a relatively compact set with respect to the uniform topology.
(3) $f(x)$ is the uniform limit of a sequence of (generalized) trigonometric polynomials $P_m(x) = \sum_{n=1}^{l} \alpha_n \exp i\lambda_n x,\ \lambda_n \in \mathbb{R}$.

The definition of almost periodicity for the continuous functions cannot carry over to generalized functions. Instead, the equivalent statements 2 or 3 can be used to define almost periodic generalized functions.

Schwartz [S] used (ii) to define almost periodic distributions in the sense of Stepanoff and showed that the following statements are equivalent for any bounded distribution T:

(1) T is almost periodic.
(2) T is the finite sum of derivatives of functions in C_{ap}.
(3) $T * \varphi \in C_{ap}$ for all $\varphi \in \mathcal{D}$.

Here, C_{ap} is the space of almost periodic functions.

Cioranescu [C2] used (iii) instead to define almost periodic non-quasianalytic ultradistributions of Beurling type and showed that the following statements are equivalent for any bounded ultradistribution T:

(1) T is almost periodic.
(2) $T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{D}^{(M_p)}$ which is the space of ultradifferentiable functions of class (M_p) in [K].

(3) There are two functions f, $g \in C_{ap}$ and an ultradifferential operator P of class (M_p) such that $T = P(D^2)f + g$.

In this section we generalize the above results of Schwartz and Cioranescu to the case of hyperfunctions and show that the following statements are equivalent for any bounded hyperfunction T:

(1) T is almost periodic.
(2) $T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{F}$ which is the Sato space defined in Section 2.
(3) There exist two functions f and g belonging to C_{ap} and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(D^2)f + g$.
(4) The Gauss transform $u(x,t) = (T * E)(x,t)$ of T is almost periodic for each $t > 0$ where $E(x,t)$ is the heat kernel.

For the proof we apply the characterization of bounded hyperfunctions in Chung-Kim-Lee [CKLe] and make use of the heat kernel method which represents various generalized functions as the initial values of solutions of the heat equation with suitable growth conditions as in Section 4.

Theorem 5.1 ([CKLe]). *The following statements are equivalent*:

(1) $T \in \mathcal{B}_{L^\infty}$.
(2) $T * \varphi \in L^\infty$ *for every* $\varphi \in \mathcal{F}$.
(3) *There exist two functions* f *and* g *belonging to* C_b *and an ultradifferential operator* P *of class* $\{p!^2\}$ *such that* $T = P(D^2)f + g$.
(4) *The Gauss transform* $u(x,t)$ *of* T *belongs to* $C^\infty(\mathbb{R}^2_+)$ *and satisfies the following:*
$$(\partial_t - \Delta)u(x,t) = 0 \quad in \;\; \mathbb{R}^2_+;$$
for every $\epsilon > 0$ *there exists a constant* $C > 0$ *such that*
$$\|u(x,t)\|_{L^\infty(\mathbb{R})} \le C\, e^{\epsilon/t} \quad in \;\; \mathbb{R}^2_+;$$
and $u(\,\cdot\,,t) \to T$ *as* $t \to 0^+$ *in the sense that*
$$T(\varphi) = \lim_{t\to 0^+} \int_{\mathbb{R}} u(x,t)\varphi(x)dx, \quad \varphi \in \mathcal{A}_{L^1}.$$

Here, C_b *is the space of bounded continuous functions on* $\mathbb{R}$.

Using the equivalent condition 3 we now define almost periodic hyperfunctions.

Definition 5.2 ([CCKK]). A hyperfunction $T \in \mathcal{B}_{L^\infty}$ is called *almost periodic* if T is the limit of a sequence of trigonometric polynomials $P_m(x) = \sum_{n=1}^{k(m)} \alpha_n \exp(i\lambda_n x)$ in the space $\mathcal{B}_{L^\infty}$ with respect to the strong topology, where $\lambda_n \in \mathbb{R}$ and $\alpha_n \in \mathbb{C}$ depend on m.

For the proof of our main result we need the following lemmas.

Lemma 5.3. *For any* $\varphi \in \mathcal{A}_{L^1}$, *let*
$$\varphi_t(x) = \int E(x-y,t)\varphi(y)dy, \quad t > 0.$$
Then $\varphi_t \in \mathcal{A}_{L^1}$ *for every* $t > 0$ *and* $\varphi_t \to \varphi$ *in* $\mathcal{A}_{L^1}$ *as* $t \to 0^+$.

We are now in a position to state the main result.

Theorem 5.4 ([CCKK]). *For* $T \in \mathcal{B}_{L^\infty}$ *the following statements are equivalent*:

(1) T *is almost periodic.*

(2) *$T * \varphi \in C_{ap}$ for every $\varphi \in \mathcal{F}$.*
(3) *There exist two functions f and g belonging to C_{ap} and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(D^2)f + g$.*
(4) *The Gauss transform $u(x,t)$ of T is almost periodic.*

As an application to the Dirichlet problem for the half plane for the case of hyperfunctions we state the following theorem without proof, which generalizes the result in [C2].

Theorem 5.5 ([CCKK]). *Let $T \in \mathcal{B}_{L^\infty}$ be almost periodic. Then there exists a harmonic function $u(x,y)$ in the right half-plane such that*

(1) *for every $x > 0$, the function $y \to u(x,y)$ is almost periodic;*
(2) *$u(x,y) \to T$ in $\mathcal{B}_{L^\infty}$ as $x \to 0$.*

6. Positive definite hyperfunctions and distributions

6.1. **Positive definite distributions.** It is well known in the theory of distributions that

(1) Every positive distribution is a measure, i.e., $u(\varphi) \geq 0$ for every $\varphi \geq 0$,
(2) Every positive tempered distribution is a tempered measure, i.e., $u(\varphi * \varphi^*) \geq 0$ for any $\varphi \geq 0$.
(3) (Bochner–Schwartz) Every positive definite (tempered) distribution is the Fourier transform of a positive tempered measure.

Recall that a generalized function u is said to be *positive* if $u(\varphi) \geq 0$ for any nonnegative test function φ and *positive definite* if $u(\varphi * \tilde{\varphi})) \geq 0$ for any nonnegative test function φ, where $\tilde{\varphi}(x) = \varphi(-x)$. Also, a positive measure μ is said to be tempered if there exists $p > 0$ satisfying $\int (1 + |x|^2)^{-p} d\mu < \infty$.

More precisely, S. Bochner proved the following theorem.

Theorem 6.1. (Bochner) *Let f be a continuous function in $\mathbb{R}^n$. Then the following statements are equivalent*:

(1) *f is positive definite, that is, for any $x_1, \ldots, x_m \in \mathbb{R}^n$ and for any complex numbers $\zeta_1, \ldots, \zeta_m$*

$$\sum_{j,k=1}^{m} f(x_j - x_k)\zeta_j \bar{\zeta_k} \geq 0.$$

(2) *f is the Fourier transform of a positive finite measure μ, i.e.,*

$$f(x) = \int e^{-i\xi \cdot x}\, d\mu(\xi).$$

(3) *For any C^∞ function φ with compact support*

$$\iint \overline{f(x-y)}\varphi(x)\overline{\varphi(y)} = \langle f, \varphi * \varphi^* \rangle \geq 0$$

where $\varphi^*(x) = \overline{\varphi(-x)}$.

The definition (6.1) of the positive definiteness for the continuous functions cannot carry over to generalized functions. Instead, the equivalent definition (6.2) will be used to define the positive definiteness for the space of generalized functions, which can be represented as a dual space of test functions.

Thus the above Bochner theorem was generalized by L. Schwartz to the space of distributions and tempered distributions, which are the dual spaces of the spaces C_c^∞ and the Schwartz space $\mathcal{S}$ as follows.

Theorem 6.2 (Bochner–Schwartz, ([S, GV])). (1) *Every positive definite distribution is the Fourier transform of a positive tempered measure, and vice versa.*

(2) *Every positive definite tempered distribution is the Fourier transform of a positive tempered measure, and vice versa.*

6.2. **Positive definite Fourier hyperfunctions.** We now generalize the above theorems to the generalized functions including (Fourier) hyperfunctions, and furthermore *Aronszajn traces* (initial values) of analytic solutions of the heat equations.

(1) Every positive hyperfunction is a measure.
(2) Every positive Aronszajn trace is nothing but a measure.
(3) Every positive Fourier hyperfunction is an infra-exponentially tempered measure.
(4) (Bochner–Schwartz) Every positive definite Fourier hyperfunction is the Fourier transform of a positive and infra-exponentially tempered measure.

Here, a positive measure μ is said to be *infra-exponentially tempered* if for every $\epsilon \geq 0$ $\int e^{-\epsilon|x|}\, d\mu < \infty$.

We state (3) more precisely which is the Bochner–Schwartz theorem for Fourier hyperfunctions.

Theorem 6.3. *Every positive definite Fourier hyperfunction is the Fourier transform of a positive infra-exponentially tempered measure μ in the sense that*

$$\mu(\varphi) = \int \hat{\varphi}(\xi) d\mu(\xi), \quad \varphi \in \mathcal{F}, \tag{6.1}$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of $\varphi(x)$.

Conversely, the functional u defined by (6.1) is a positive definite Fourier hyperfunction.

6.3. **Positive definite hyperfunctions.** In this Subsection we state our version of Bochner-Schwartz theorem for hyperfunctions.

Recall that the heat kernel $E(x, t)$ belongs to the Sato space $\mathcal{F}$ for each $t > 0$. Thus the defining function or Gauss transform

$$U(x, t) = u_y(E(x - y, t), \quad x \in \mathbb{R}^n, \ t > 0$$

is well defined for an analytic functional $u \in \mathcal{A}'(K)$ or a Fourier hyperfunction $u \in \mathcal{F}$. Intuitively, the defining function $U(x, t)$ is the convolution of u and the heat kernel $E(x, t)$, i.e., $U(x, t) = (u *_x E)(x, t)$.

In order to define positive definite hyperfunctions we note that a hyperfunction is defined locally as analytic functionals, in other word, that the space $\mathcal{B}$ of hyperfunctions is defined locally as the space of analytic functionals which is the dual space of analytic functionals, but not globally. To overcome this difficulty we apply the heat kernel method of T. Matsuzawa once again as the case of periodic hyperfunctions as in [M1], [KCK] and [CsK1]. We make use of the representations of the generalized functions including distributions, hyperfunctions and Fourier hyperfunctions as the initial values of the solutions of the heat equation, and then *define positive definite hyperfunctions* in terms of the defining function as follows:

We are now in a position to define the positive definite hyperfunction in terms of the defining function and the growth condition.

Definition 6.4 ([CCK3]). A hyperfunction u is *positive definite* if there exists a defining function $U(x,t)$ of u is a positive definite function for each $t>0$, that is,

$$\sum_{j,k=1}^{n} U(x_j - x_k, t)\zeta_j \bar{\zeta}_k \geq 0$$

for every $x_1, \ldots, x_n \in \mathbb{R}^n$, $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and for each $t>0$.

To justify the above definition for the positive definiteness of the hyperfunctions we prove the equivalence of our new definitions and the original definitions of the positive definiteness for

(1) the continuous functions
(2) (tempered) distributions
(3) Fourier hyperfunctions

respectively in [CsK2]. As this natural definition of positive definite hyperfunctions is given we can easily prove the following the Bochner–Schwartz theorem for hyperfunctions.

Theorem 6.5 ([CCK3]). *The following conditions are equivalent*:

(1) *u is a positive definite hyperfunction.*
(2) *u is a positive definite Fourier hyperfunction.*

We sum up the results in this section for distributions and hyperfunctions respectively as follows.

Theorem 6.6. *The following conditions are equivalent*:

(1) *u is a positive definite distribution.*
(2) *u is a positive definite tempered distribution.*
(3) *u is the Fourier transform of a positive tempered measure.*
(4) *The defining function $U(\cdot,t)$ of $u \in \mathcal{D}'$ is a positive definite function for each $t>0$.*
(5) *The defining function $U(\cdot,t)$ of $u \in \mathcal{S}'$ is a positive definite function for each $t>0$.*

As a parallel result of the Bochner–Schwartz theorem we have the following:

Theorem 6.7. *The following conditions are equivalent:*

(1) *u is a positive definite hyperfunction.*
(2) *u is a positive definite Fourier hyperfunction.*
(3) *u is the Fourier transform of a positive infra-exponentially tempered measure.*
(4) *The defining function $U(\cdot,t)$ of $u \in \mathcal{F}'$ is a positive definite function for each $t>0$.*

7. Conditionally positive definite (Fourier) hyperfunctions and distributions

7.1. **Conditionally positive definite distributions.** On the other direction the concept of positive definite (generalized) functions has been generalized to the conditionally positive definite functions and distributions which arise in the theory of generalized random process.

To introduce the conditionally positive definite generalized functions, note that for any continuous function $f(x)$ on $\mathbb{R}^n$ the following conditions are equivalent:

(1) $\exp(cf(x))$ is positive definite for all $c > 0$.
(2) For any $x_1, x_2, \cdots, x_m \in \mathbb{R}^n$ and any complex numbers such that $\zeta_1 + \zeta_2 + \cdots + \zeta_m = 0$,

$$\sum_{j,k=1}^{m} f(x_j - x_k)\zeta_j\overline{\zeta_k} \ \geq \ 0 \tag{7.1}$$

(3) For any C^∞ function φ with compact support

$$\langle f, (\partial_j\varphi) * (\partial_j\varphi)^* \rangle \geq 0 \tag{7.2}$$

for all $j = 1, \cdots, n$.

Generalizing the partial derivatives involved in (7.2) to any linear homogeneous constant coefficient differential operators of order s the conditionally positive definite distributions is defined in [GV].

Definition 7.1 ([GV]). A distribution $u \in \mathcal{D}$ is said to be *conditionally positive definite* of order s if

$$\langle u, (D\varphi) * (D\varphi)^* \rangle \geq 0 \tag{7.3}$$

for all test functions $\varphi \in \mathcal{D}$ and all constant coefficient differential operators

$$D = \sum_{|\alpha|=s} a_\alpha \partial^\alpha. \tag{7.4}$$

Gel'fand–Vilenkin proved the following Bochner–Schwartz theorem for conditionally positive distributions in their famous treatise "Generalized Functions, vol. 4" as follows.

Theorem 7.2 ([GV]). *Every conditionally positive definite* (*tempered*) *distribution u of order s can be expressed as*

$$\langle u, \varphi \rangle = \int_{\mathbb{R}_0^n} \left(\hat{\varphi}(x) - \alpha(x) \sum_{|k|=0}^{2s-1} \frac{\hat{\varphi}^{(k)}(0)}{k!} x^k \right) d\mu(x) + \sum_{|k|=0}^{2s} a_k \frac{\hat{\varphi}^{(k)}(0)}{k!}, \tag{7.5}$$

where

(i) *μ is a positive measure on $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$ satisfying*

$$\int_{0<|x|\leq 1} |x|^{2s} d\mu(x) < \infty \quad \textit{and} \quad \int_{|x|\geq 1} (1+|x|)^{2-p} d\mu(x) < \infty$$

for some $p \geq 0$;

(ii) the function $\alpha(x) \in \widehat{C}_c^\infty$ and $\alpha(x) - 1$ has a zero of order $2s + 1$ at $x = 0$;

(iii) a_k's, $|k| = 2s$ are the complex numbers such that

$$\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j \geq 0$$

for all complex numbers $\xi_j \in \mathbb{C}$, $|j| = s$ and a_k's, $|k| \le 2s-1$, are certain complex numbers.

7.2. **Conditionally positive definite Fourier hyperfunctions.** In this subsection we first characterize the conditionally positive Fourier hyperfunctions to obtain the Bochner–Schwartz type theorem for conditionally positive definite Fourier hyperfunctions.

Definition 7.3. A Fourier hyperfunction u is said to be *conditionally positive definite* of order s if
$$\langle u, (P\varphi) * (P\varphi)^* \rangle \ge 0$$
for all test functions φ and all constant coefficient differential operators
$$P = \sum_{|\alpha|=s} a_\alpha \partial^\alpha.$$

Since the space $\mathcal{F}$ is invariant under the Fourier transformation, the inequality (7.3) is equivalent to the following inequality for the Fourier transform $F = \hat{u}$
$$\langle P\bar{P}F, \psi\bar{\psi} \rangle \ge 0$$
for all homogeneous polynomial P of degree s and all test functions $\psi \in \mathcal{F}$. We call such a generalized function F a *conditionally positive Fourier hyperfunction of order s*.

For this we need the following lemma.

Lemma 7.4. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be the subspaces of $\mathcal{S}_a^b(\mathbb{R}^n)$ such that*
$$\mathcal{Q}_1 = \{\varphi \in \mathcal{S}_a^b \mid \varphi = \sum_{|k|=m} x^k \psi_k(x),\, \psi_k \in \mathcal{S}_a^b\}$$
$$\mathcal{Q}_2 = \{\varphi \in \mathcal{S}_a^b \mid \varphi^{(k)}(0) = 0,\, |k| < m\}$$
Then $\mathcal{Q}_1$ is dense in $\mathcal{Q}_2$. In particular if $n = 1$, then $\mathcal{Q}_1 = \mathcal{Q}_2$.

Hereafter we put
$$\omega(x) = e^{-|x|^2} \sum_{|k|=0}^{s} \frac{1}{k!} x^{2k}. \tag{7.6}$$
It is easy to see that $\omega(x) \in \mathcal{F}$ and $\omega(x) - 1$ has a zero of order $2s+1$ at $x = 0$, i. e., $\omega(0) = 1$ and $\omega^{(k)}(0) = 0$ for all $0 < |k| \le 2s$.

We are now in a position to state the main theorems.

Theorem 7.5. *Every conditionally positive Fourier hyperfunction F of order s can be expressed as*
$$\langle F, \varphi \rangle = \int_{\mathbb{R}_0^n} \left(\varphi(x) - \omega(x) \sum_{|k|=0}^{2s-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right) d\mu(x) + \sum_{|k|=0}^{2s} a_k \frac{\varphi^{(k)}(0)}{k!},$$
where

(i) *μ is a positive measure on $\mathbb{R}_0^n$ such that*
$$\int_{0<|x|\le 1} |x|^{2s} d\mu(x) < \infty, \quad \int_{|x|\ge 1} e^{-\varepsilon|x|} d\mu(x) < \infty$$
for every $\varepsilon > 0$;

(ii) a_k's, $|k| = 2s$ are the complex numbers such that

$$\sum_{|i|=|j|=s} a_{i+j}\xi_i\bar{\xi}_j \geq 0$$

for all complex numbers $\xi_j \in \mathbb{C}$, $|j| = s$ and a_k's, $|k| \leq 2s-1$, are certain complex numbers.

We can show that the converse of Theorem 7.5 holds true.

¿From the Parseval relation $(2\pi)^n\langle F, \varphi\rangle = \langle \hat{F}, \hat{\varphi}\rangle$ we have the following Bochner–Schwartz type theorem for conditionally positive definite Fourier hyperfunctions.

Theorem 7.6. *The following conditions are equivalent.*

(1) *A generalized function u is a conditionally positive definite Fourier hyperfunction of order s.*

(2) *u can be expressed as*

$$\langle u, \varphi\rangle = \int_{\mathbb{R}_0^n}\left(\hat{\varphi}(x) - \omega(x)\sum_{|k|=0}^{2s-1}\frac{\hat{\varphi}^{(k)}(0)}{k!}x^k\right)d\mu(x) + \sum_{|k|=0}^{2s} a_k\frac{\hat{\varphi}^{(k)}(0)}{k!}, \tag{7.7}$$

where

(i) *μ is a positive measure on $\mathbb{R}_0^n$ such that*

$$\int_{0<|x|\leq 1}|x|^{2s}d\mu(x) < \infty, \quad \text{and} \quad \int_{|x|\geq 1}e^{-\varepsilon|x|}d\mu(x) < \infty \tag{7.8}$$

for every $\varepsilon > 0$;

(ii) a_k's, $|k| = 2s$ are the complex numbers such that

$$\sum_{|i|=j|=s} a_{i+j}\xi_i\bar{\xi}_j \geq 0$$

for all complex numbers $\xi_j \in \mathbb{C}, |j| = s$ and a_k's, $|k| \leq 2s-1$ are certain complex numbers.

Note that a positive definite Fourier hyperfunction u is a conditionally positive definite Fourier hyperfunction of order $s = 0$. Thus from Theorem 7.6 we have

$$\langle u, \varphi\rangle = \int_{\mathbb{R}^n\setminus\{0\}}\hat{\varphi}(x)\mu(x) + a_0\hat{\varphi}(0).$$

Here μ is a positive measure on $\mathbb{R}_0^n$ such that $\int_{\mathbb{R}_0^n} e^{-\varepsilon|x|}d\mu(x) < \infty$ for every $\varepsilon > 0$ and $a_0 \geq 0$. Therefore, if we define $\mu(0) = a_0$ we have the Bochner–Schwartz theorem for Fourier hyperfunctions as in [CsK2].

7.3. **Conditionally positive definite hyperfunctions.** Recall that a hyperfunction $u \in \mathcal{B}(\mathbb{R}^n)$ is said to be *positive definite* if there exists a Gauss transform $U(x,t)$ of u such that $U(x,t)$ is positive definite for each $t > 0$. In accordance with the definition of positive definite hyperfunctions we define a conditionally positive definite hyperfunction.

Definition 7.7. A hyperfunction u is said to be *conditionally positive definite* if there exists a Gauss transform $U(x,t)$ of u such that $P\bar{P}U(\cdot\,,t)$ is positive definite for all differential operators P of the form (7.4) and for each $t > 0$.

To justify the above definition we state and prove some analogous results in the spaces of distributions, tempered distributions and Fourier hyperfunctions.

Theorem 7.8. *The following conditions are equivalent.*

(1) *u is a conditionally positive definite tempered distribution of order s.*
(2) *The Gauss transform $U(x,t)$ of u has the property that $P\bar{P}U(\cdot\,,t)$ is a positive definite function for each $t>0$ and for all differential operators P of the form (7.4).*

Theorem 7.9. *The following conditions are equivalent.*

(1) *u is a conditionally positive definite distribution of order s.*
(2) *There exists a Gauss transform $U(x,t)$ of u such that $U(x,t)$ has the property that $P\bar{P}U(\cdot\,,t)$ is a positive definite function for each $t>0$ and for all differential operator P of the form (7.4)*

By the same method as in Theorem 7.8 we obtain the similar result for Fourier hyperfunctions.

Theorem 7.10. *The following conditions are equivalent.*

(1) *u is a conditionally positive definite Fourier hyperfunction of order s.*
(2) *The Gauss transform $U(x,t)$ of u has the property that $P\bar{P}U(\cdot,t)$ is a positive definite function for each $t>0$ and all differential operator P of the form (7.4).*

Now we state our main results.

Theorem 7.11. *The following conditions are equivalent for $u \in \mathcal{B}(\mathbb{R}^n)$.*

(1) *u is a conditionally positive definite hyperfunction of order s.*
(2) *u is a conditionally positive definite Fourier hyperfunction of order s.*

Combining Theorem 7.8 and Theorem 7.9 we have the following result.

Corollary 7.12. *The following conditions are equivalent.*

(1) *u is a conditionally positive definite distribution of order s .*
(2) *u is a conditionally positive definite tempered distribution of order s*
(3) *The Gauss transform $U(x,t)$ of u satisfies the growth condition ; there exist positive constants C and N such that*

$$|U(x,t)| \le C(1+|x|)^{2s}\, t^{-N}$$

and $P\bar{P}U(\cdot\,,t)$ is positive definite function for each $t>0$ and all differential operators P of the form (7.4).
(4) *u can be expressed as in (7.5) of Theorem 7.2.*

Combining Theorem 7.10 and Theorem 7.11 we have the following.

Corollary 7.13. *The following conditions are equivalent*

(1) *u is a conditionally positive definite hyperfunction of order s .*
(2) *u is a conditionally positive definite Fourier hyperfunction of order s.*
(3) *The Gauss transform $U(x,t)$ of u satisfies the following growth condition ; for every $\varepsilon>0$, there exists a constant C_ε such that*

$$|U(x,t)| \le C_\varepsilon(1+|x|)^{2s} \exp(\varepsilon/t)$$

and $P\bar{P}U(\cdot\,,t)$ is positive definite function for each $t>0$ and all differential operators P of the form (7.4).

(4) *u can be expressed as in (7.7) where the measure μ in (7.7) satisfies the growth conditions (7.8) and $\omega(x)$ in (7.7) is given by (7.6).*

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