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DYNAMICAL SYSTEMS OVER ALGEBRAIC STRUCTURES

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1. Four algebraic structures

- **topological groups**: A topological group is a locally compact Hausdorff topological space G with an element e and two continuous maps $. : G\times G \to G$ and $-1 : G \to G$ satisfying:
  $(G1)$ $(x.y).z = x.(y.z)$,
  $(G2)$ $x.e = e.x = x$,
  $(G3)$ $x.x^{-1} = x^{-1}.x = e$,
  for each $x, y, z \in G$.

- **inverse semigroups**: A unital inverse semigroup is a discrete space S with an element e and two operations $. : S \times S \to S$ and $* : S \to S$ satisfying
  $(S1)$ $(x.y).z = x.(y.z)$,
  $(S2)$ $x.e = e.x = x$,
  $(S3)$ $x.y.x = x$, $y.x.y = y$ if and only if $y = x^*$,
  for each $x, y, z \in S$. Note that the last item implies that $x^{**} = x$. The set $E_S = \{ss^* : s \in S\}$ of idempotents is a commutative subsemigroup of S.

- **topological hypergroups**: A topological hypergroup is a locally compact Hausdorff space K with an element e and two operations $* : M(K) \times M(K) \to M(K)$ ($M(K)$ =Banach space of all bounded Borel measures on K with the variation norm) and $^- : K \to K$ satisfying

$(H1)$ $M(K)$ is an algebra under $*$,
$(H2)$ $\delta_x * \delta_y$ is a probability measure,
$(H3)$ The map $(x,y) \mapsto \delta_x * \delta_y$ of $K \times K \to M(K)$ is continuous in the weak* topology,
$(H4)$ The map $(x,y) \mapsto supp(\delta_x * \delta_y)$ of $K \times K \to Comp(K)$ is continuous in the Michael topology,
$(H5)$ $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$,
$(H6)$ $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$,
$(H7)$ e is in the support of $\delta_x * \delta_y$ if and only if $x = \bar{y}$,
for each $x, y \in K$.

- **topological groupoids**: A topological groupoid is a locally compact space $\mathcal{G}$ with a distinguished set $\mathcal{G}^{(0)}$ and four continuous operations $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$, $-1 : \mathcal{G} \to \mathcal{G}$, and $. : \mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ ($\mathcal{G}^{(2)}$ = the set of all pairs $(x,y) \in \mathcal{G} \times \mathcal{G}$ with $s(x) = r(y)$) satisfying
$(\mathcal{G}1)$ If $(x,y), (y,z) \in \mathcal{G}^{(2)}$ then $(x.y).z = x.(y.z)$,
$(\mathcal{G}2)$ If $(x,y) \in \mathcal{G}^{(2)}$, then $s(x.y) = s(y)$, $r(x.y) = r(x)$,
$(\mathcal{G}3)$ If $u \in \mathcal{G}^{(0)}$ then $s(u) = r(u) = u$,
$(\mathcal{G}4)$ $x.s(x) = r(x).x = x$,
$(\mathcal{G}5)$ $x.x^{-1} = r(x)$, $x^{-1}.x = s(x)$,
$(\mathcal{G}6)$ $\mathcal{G}^{(0)}$ is Hausdorff,
$(\mathcal{G}7)$ $x \in \mathcal{G}$ has a Hausdorff neighborhood,
for each $x, y, z, u \in \mathcal{G}$.

2. Universal Examples

- **topological groups**:

  - Let X be a non-empty topological space and $G = Aut(X)$ be the set of all continuous bijections from X onto X, then G is a topological group under the composition of functions and topology of uniform convergence on compact sets.

  - Let H be a Hilbert space and $G = \mathcal{U}(H)$ be the set of all unitary operators on H, then G is a topological group under the composition of operators and WOT.

- **inverse semigroups**:

  - Let X be a non-empty set and $S = End(X)$ be the set of all injections from X into X, then S is a unital inverse semigroup under the composition of functions and inverse of functions.

  - Let H be a Hilbert space and $S = \mathcal{PI}(H)$ be the set of all partial isometries on H, then S is a unital inverse semigroup under the composition of operators and adjoint operation.

- **topological hypergroups**: I am not aware of any universal example (and I doubt that it could exist!)

- **topological groupoids**:

  - Let X be a non-empty topological space and $\mathcal{G} = C(X)$ be the set of all continuous functions from X into X, then $\mathcal{G}$ is a topological group under the range and domain maps and composition of functions ($f \circ g$ is defined only when $R_f = D_g$) and topology of uniform convergence on compact sets.

  - Let H be a Hilbert space and $\mathcal{G} = C(H)$ be the set of all closed (possibly unbounded) operators on H, then $\mathcal{G}$ is a topological groupoid under the composition of operators and WOT.

3. More Examples

- **topological groups**:

  - Abelian groups: $(\mathbb{Z}, +), (\mathbb{T}, .), (\mathbb{R}, +)$,

  - non-Abelian groups: $U(2), SU(3)$, Heisenberg $ax + b$, Fundamental group $\pi_1(\mathcal{M})$.

- **inverse semigroups**:

  - $(\mathbb{N}, +)$ (with $n^* = n$),

  - Cuntz semigroup S_n generated by n partial isometries $s_1, \ldots, s_n \in B(H)$ satisfying $s_i^* s_i = I \, (1 \le i \le n)$ and $\sum_{i=1}^n s_i s_i^* = I$.

- **topological hypergroups**:

  - finite hypergroup $\mathbb{Z}_\theta(2) = \{0, 1\}$ with

$$\delta_1 * \delta_1 = \theta\delta_0 + (1 - \theta)\delta_1 \, (0 \le \theta \le 1),$$

  - commutative infinite join hypergroup $\mathbb{N}$ with $\delta_n * \delta_m = \delta_{max(n,m)}$ $(n \neq m)$ and

$$\delta_n * \delta_n = b_n \prod_{k=1}^{n-1} \frac{b_k}{b^k + 1}\delta_0 + \sum_{j=1}^{n-1} \frac{b_n}{b_j} \prod_{k=1}^{n-1} \frac{b_k}{b^k + 1}\delta_j + (1 - b_n)\delta_n.$$

  - commutative class hypergroup K(G)=representatives of conjugacy classes of a finite group G, for instance $K(S_3) = \{c_0, c_1, c_2\}$ with

$$\delta_{c_1} * \delta_{c_1} = \frac{1}{3}\delta_{c_0} + \frac{2}{3}\delta_{c_2}, \; \delta_{c_1} * \delta_{c_2} = \delta_{c_1}, \; \delta_{c_2} * \delta_{c_2} = \frac{1}{2}\delta_{c_0} + \frac{1}{2}\delta_{c_2}.$$

  - discrete dual hypergroup $\hat{G}$=the dual space of a compact group G with

$$\delta_\pi * \delta_\rho = \sum_{\tau \in \pi \otimes \rho} \frac{dim\tau}{dim\pi . dim\rho} m_{\tau,\pi,\rho}\delta_\tau,$$

for instance $\hat{S}_3 = \{\chi_0, \chi_1, \chi_2\}$ with

$$\delta_{\chi_1} * \delta_{\chi_1} = \delta_{\chi_0},\ \delta_{\chi_1} * \delta_{\chi_2} = \delta_{\chi_2},\ \delta_{\chi_2} * \delta_{\chi_2} = \frac{1}{4}\delta_{\chi_0} + \frac{1}{4}\delta_{\chi_1} + \frac{1}{2}\delta_{\chi_2}.$$

- **topological groupoids**:

- group actions: If a (topological) group G acts on a set (topological space) X, then $\mathcal{G} = X \times G$ is a (topological) groupoid with $\mathcal{G}^{(0)} = X$ and $r(x,g) = x$, $s(x,g) = x.g$, $(x,g).(x.g,h) = (x,gh)$.

- vector bundles: If $\pi : \mathcal{E} \to \mathcal{M}$ is a (smooth) vector bundle over a (smooth) manifold $\mathcal{M}$, then $\mathcal{E}$ is a groupoid with $\mathcal{E}^{(0)} = \mathcal{M}$, and $s(e) = r(e) = \pi(e)$, $e.f = e + f$.

- Connes holonomy groupoid: For the foliated space $(\mathcal{M}, \mathcal{F})$ the set $\mathcal{G}$ of triples $(x, y, [\alpha])$ where x and y lie on the same leaf L and $[\alpha]$ is the holonomy class of a piecewise smooth curve α from x to y in L, is a (smooth) groupoid with $\mathcal{G}^{(0)} = \mathcal{M}$ and $r(x,y,[\alpha]) = y, s(x,y,[\alpha]) = x$, $(y,z,[\beta]).(x,y,[\alpha]) = (x,z,[\alpha.\beta])$, where $\alpha.\beta$ is the concatenation of the curves α and β. This groupoid is not Hausdorff in general.

- tangent groupoid: Let $\mathcal{M}$ be a smooth manifold with tangent bundle $T\mathcal{M}$, then $\mathcal{G} = (\mathcal{M} \times \mathcal{M} \times (0,1]) \cup T\mathcal{M}$ is a smooth groupoid.

- Cuntz groupoid $\mathcal{O}_n$ [17].

4. REPRESENTATIONS

- **topological groups**: A representation of a topological group G is a pair $\{\pi, \mathcal{H}_\pi\}$ where $\mathcal{H}_\pi$ is a Hilbert space and $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a continuous group homomorphism in WOT. An example of particular interest is the left regular representation $\pi_L : G \to \mathcal{B}(L^2(G))$ defined by

$$\pi_L(x)\xi(y) = \xi(x^{-1}y) \quad (x, y \in G, \xi \in L^2(G)).$$

This is a faithful representation.

- (**Schur**) Irreducible representations of compact groups are finite dimensional.

- **inverse semigroups**: A representation of an inverse semigroup S is a pair $\{\pi, \mathcal{H}\}$ where $\mathcal{H}$ is a Hilbert space with subspaces $\mathcal{H}_e$ for $e \in E_S$ such that for each $s \in S$, $\pi(s) : \mathcal{H}_{ss^*} \to \mathcal{H}_{s*s}$ is a partial isometry [20]. The left regular representation $\pi_L : S \to \mathcal{B}(\ell^2(S))$ is defined by

$$\pi_L(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq y^*y \\ 0 & \text{otherwise} \end{cases} \quad (\xi \in \ell^2(S), x, y \in S),$$

and it is faithful [22].

- I don't know if irreducible representations of finite inverse semigroups are finite dimensional.

- **topological hypergroups**: A representation of a topological hypergroup K is a $*$-representation $\{\pi, \mathcal{H}\}$ of the Banach $*$-algebra $M(K)$ such that $\pi(\delta_e) = I$ and, for each $\xi, \zeta \in \mathcal{H}$, the map $\mu \mapsto \langle \pi(\mu)\zeta, \xi \rangle$ is continuous on $M_+(K)$ with weak topology. Left regular representation has to be defined by a convolution operator. They are not well studied.

  - Irreducible representations of compact hypergroups are finite dimensional [8].

- **topological groupoids**: A representation of a topological groupoid $\mathcal{G}$ is a triple $(\pi, \mathcal{H}_\pi, \mu_\pi)$, where $\mathcal{H}_\pi = \{\mathcal{H}_u^\pi\}$ is a bundle of Hilbert spaces, μ_π is a quasi-invariant measure on X (with associated measures ν, ν^{-1}, ν^2, and ν_0) such that
  $(i)\, \pi(x) \in \mathcal{B}(\mathcal{H}^\pi_{s(x)}, \mathcal{H}^\pi_{r(x)}) \quad (x \in \mathcal{G})$,
  $(ii)\, \pi(u) = id_u : \mathcal{H}_u^\pi \to \mathcal{H}_u^\pi \quad (u \in X)$,
  $(iii)\, \pi(xy) = \pi(x)\pi(y)$ for ν^2-a.e. $(x, y) \in \mathcal{G}^{(2)}$,
  $(iv)\, \pi(x^{-1}) = \pi(x)^{-1}$ for ν-a.e. $x \in \mathcal{G}$,
  $(v)\, x \mapsto < \pi(x)\xi(s(x)), \eta(r(x)) >$ is ν-measurable on $\mathcal{G}$ for each $\xi, \eta \in L^2(\mathcal{G}^{(0)}, \mathcal{H}_\pi, \mu_\pi)$ (see section 7 for a definition of this L^2-space.)
  If the items $(i) - (iv)$ happen everywhere (instead of almost everywhere) and the map in (v) is continuous (instead of measurable), $(\pi, \mathcal{H}_\pi, \mu_\pi)$ is called a continuous representation.
  The left regular representation $\pi_L : \mathcal{G} \to \mathcal{B}(L^2(\mathcal{G}))$ is defined by

$$\pi_L(x)\xi(y) = \xi(x^{-1}y) \quad (x \in \mathcal{G}, y \in \mathcal{G}^{s(x)}, \xi \in L^2(\mathcal{G}^{s(x)})).$$

  Note that $L^2(\mathcal{G})$ is defined in the bundle sense. This is a faithful representation.

  - Irreducible continuous representations of compact groupoids have finite dimensional fibers [2] (but they are not finite dimensional in general.)

5. DUALITY

- **topological groups**:

  - The set $\hat{G}$ of unitary equivalence classes of irreducible representations of a topological group G is locally compact T_0 topological space (not necessarily Hausdorff). $\hat{G}$ is compact (discrete) if G is discrete (compact).

  - (**Pontryagin**) The set $\hat{G}$ of characters of an Abelian topological group G is an Abelian topological group under pointwise multiplication and induced weak* topology. $(\hat{G})\hat{} \simeq G$ (Pontryagin duality). $\hat{G}$ is compact (discrete) if and only if G is discrete (compact).

  - (**Tannaka-Krein**) When G is compact, one can recover G from $\hat{G}$ (only using the data in $\hat{G}$ we construct the Tannaka group $T(G)$ and show that $T(G) \simeq G$.

- **inverse semigroups**:

  - For non-commutative inverse semigroup S, the dual space $\hat{S}$ is not so much studied.

  - The set $\hat{S}$ of semi-characters of a unital commutative inverse semigroup S is a commutative compact semigroup under pointwise multiplication and induced weak* topology. $(\hat{S})\hat{} \not\cong S$ in general.

- **topological hypergroups**:

  - For non-commutative hypergroup K, the dual space $\hat{K}$ is somehow studied [8].

  - The set $\hat{K}$ of characters of a commutative hypergroup K is a not a hypergroup in general. Even if $\hat{K}$ is a hypergroup, $(\hat{K})\hat{} \not\cong K$ in general (unless K is discrete). If K is compact (discrete) then $\hat{K}$ is discrete (compact).

- **topological groupoids**:

  - For topological groupoid $\mathcal{G}$, the dual space $\hat{\mathcal{G}}$ is not studied at all.

  - For a compact groupoid $\mathcal{G}$, it is shown that $\mathcal{G}$ could be recovered from $\hat{\mathcal{G}}$ [2] (by constructing the Tannaka groupoid of $\mathcal{G}$.)

  - A commutative groupoid $\mathcal{G}$ is a bundle of Abelian groups and so not very interesting.

  - **(Peter-Weyl Theorem)** Let $\mathcal{G}$ be a compact groupoid, then for each $u, v \in X$, $\mathcal{E}^{\pi}_{u,v}$ is dense in $C(\mathcal{G}^{v}_{u})$ and

$$L^2(\mathcal{G}^v_u, \lambda^v_u) = \bigoplus_{\pi \in \hat{\mathcal{G}}} \mathcal{E}^{\pi}_{u,v},$$

and

$$\{\sqrt{d^{\pi}_{u}}\pi^{ij}_{u,v} : \pi \in \hat{\mathcal{G}},\ 1 \le i \le d^{\pi}_{v},\ 1 \le j \le d^{\pi}_{u}\},$$

is an orthonormal basis for $L^2(\mathcal{G}^v_u, \lambda^v_u)$. Each $\pi \in \hat{\mathcal{G}}$ occurs in the right and left regular representation of $\mathcal{G}$ over $L^2(\mathcal{G}^v_u, \lambda^v_u)$ with multiplicity d^{π}_{u} [2].

6. L^1-ALGEBRAS

- **topological groups**: For a topological group G with left Haar measure λ,

$$L^1(G) = \{f : G \to \mathbb{C} : f \text{ Borel measurable}, \|f\|_1 = \int_G |f| d\lambda < \infty\}$$

is a Banach $*$-algebra with respect to the following operations and above norm

$$f * g(x) = \int_G f(y) g(y^{-1}x) d\lambda(y),\ f^*(x) = \Delta(x^{-1})\bar{f}(x^{-1}).$$

  - $L^1(G)$ has a bounded approximate identity. It is amenable (as a Banach algebra) iff G is amenable (as a group.)

- **inverse semigroups**: For an inverse semigroup S,

$$\ell^1(S) = \{f : S \to \mathbb{C} : \|f\|_1 = \sum_{s \in S} |f(s)| < \infty\}$$

is a Banach $*$-algebra with respect to the following operations and above norm

$$f * g(x) = \sum_{st=x} f(s)g(t), \ f^*(x) = \bar{f}(x^*).$$

- $\ell^1(S)$ fails to have a bounded approximate identity in general. If it is amenable (as a Banach algebra) then E_S is finite. In this case, amenability of $\ell^1(S)$ is equivalent to amenability of all maximal subgroups of S [14]. On the other hand $\ell^1(S)$ is amenable as a Banach $\ell^1(E_S)$-module iff S is amenable [3].

- **topological hypergroups**: They don't have an L^1-algebra in general. Some work has been done for measured hypergroups [8].

- **topological groupoids**: There is a bundle version of $L^1(\mathcal{G})$, but it is more convenient to work with continuous functions with compact support in this case. As $\mathcal{G}$ is not Hausdorff in general, one should define this algebra carefully. Indeed $C_c(\mathcal{G})$ is defined as the span of the set of functions $f : \mathcal{G} \to \mathbb{C}$ which are continuous of compact support on an open Hausdorff subset of $\mathcal{G}$ and zero outside. It is a normed $*$-algebra with respect to the following operations and norm

$$f * g(x) = \int_{\mathcal{G}^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y), \ f^*(x) = \bar{f}(x^{-1}),$$

$$\|f\|_I = max\big(\sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^u} |f| d\lambda^u, \ \sup_{u \in \mathcal{G}^{(0)}} \int_{\mathcal{G}_u} |f| d\lambda_u \big).$$

- Note that elements of $C_c(\mathcal{G})$ might fail to be continuous on whole of $\mathcal{G}$. Also $C_c(\mathcal{G})$ isn't closed under pointwise multiplication and taking absolute values in general. It has a bounded approximate identity, when $\mathcal{G}$ is r-discrete [14].

7. C^*-ALGEBRAS

- **topological groups**: For $\pi \in Rep(G)$ and $f \in L^1(G)$ put

$$\tilde{\pi}(f) = \int_G f(x)\pi(x)d\lambda(x),$$

where the right hand side is a Bochner integral, and define

$$\|f\| = \sup_{\pi} \|\tilde{\pi}(f)\|, \ \|f\|_r = \|\tilde{\pi}_L(f)\|.$$

The completions of $L^1(G)$ with respect to these C^*-norms are called the full and reduced group C^*-algebras and are denoted by $C^*(G)$ and $C^*_r(G)$, respectively.

- $C_r^*(G)$ is nuclear iff G is amenable iff $C^*(G) \simeq C_r^*(G)$.

- **inverse semigroups**: For $\pi \in Rep(S)$ and $f \in \ell^1(S)$ put

$$\tilde{\pi}(f) = \sum_S f(s)\pi(s),$$

where the right hand is convergent in WOT, and define

$$\|f\| = \sup_\pi \|\tilde{\pi}(f)\|, \ \|f\|_r = \|\tilde{\pi}_L(f)\|.$$

The completions of $\ell^1(S)$ with respect to these C^*-norms are called the full and reduced semigroup C^*-algebras and are denoted by $C^*(S)$ and $C_r^*(S)$, respectively. A few examples are instructive:

- If $\mathcal{F}$ is the free monogenic inverse semigroup, then $C^*(\mathcal{F}) \subseteq \mathcal{B}(\oplus_{n=2}^{\infty}\mathbb{C}^n)$ is the C^*-algebra generated by the infinite truncated shift (and it's adjoint).

- If G is a (discrete) group with identity e, P is a subsemigroup of G containing e with $PP^{-1} = G$, and β_x is the compression of $\pi_L(x)$ to $\ell^2(P)$, then the inverse semigroup $S_{G,P}$ is studied in [13]. When $G = \mathbb{R}$ and $P = \mathbb{R} \cup \{0\}$, $C_r^*(S_{G,P})$ is the Wiener-Hopf algebra.

- If A is an $n \times n$ matrix with $0, 1$ entries and nonzero rows and columns and $s_1, \ldots, s_n$ are partial isometries on a Hilbert space $\mathcal{H}$ such that, for $p_i = s_i s_i^*$ and $q_i = s_i^* s_i$,

$$p_i p_j = 0 \, (i \neq j), \quad q_i = \sum_j A_{ij} p_j$$

then the C^*-algebra $\mathcal{O}_A$ generated by s_i's (and their adjoints) is called the Cuntz-Kreiger algbera of A. (Note that $\mathcal{O}_A$ is not independent of the Hilbert space $\mathcal{H}$ on which the partial isometries are represented.) When all entries of A are 1, $\mathcal{O}_A$ is denoted by $\mathcal{O}_n$ and is called the Cuntz algebra. One can associate an inverse semigroup S_A to A with $C^*(S_A) \simeq \mathcal{O}_A$ [14].

For any inverse semigroup S we have the following characterization of nuclearity of reduced semigroup C^*-algebra.

- $C_r^*(S)$ is nuclear iff a family of subsemigroups of S indexed by the spectrum of the commutative C^*-algebra $C^*(E_S)$ are amenable and S is hyperfinite. In this case, $C^*(S) \simeq C_r^*(S)$ [4].

- **topological hypergroups**: The full C^*-algebra of a measured hypergroup could be defined similarly. They are barely studied.

- **topological groupoids**: For $\pi \in Rep(\mathcal{G})$ and $f \in C_c(\mathcal{G})$ put

$$\langle \tilde{\pi}(f)\xi, \eta \rangle = \int_{\mathcal{G}} f(x) \langle \pi(x)\xi(s(x)), \eta(r(x)) \rangle d\nu_0(x),$$

and define

$$\|f\| = \sup_\pi \|\tilde{\pi}(f)\|, \ \|f\|_r = \|\tilde{\pi}_L(f)\|.$$

Note that $\tilde{\pi}(f)$ is an operator in $\mathcal{B}(L^2(\mathcal{G}^{(0)}, \mathcal{H}_\pi; \mu_\pi))$, where

$$L^2(\mathcal{G}^{(0)}, \mathcal{H}_\pi, \mu_\pi) = \{f : \mathcal{G}^{(0)} \to \bigcup_u \mathcal{H}_\pi^u : \int_{\mathcal{G}^{(0)}} \|f(u)\|_u^2 d\mu_\pi < \infty\}.$$

The completions of $L^1(\mathcal{G})$ with respect to these C^*-norms are called the full and reduced group C^*-algebras and are denoted by $C^*(\mathcal{G})$ and $C_r^*(\mathcal{G})$, respectively.

- If $\mathcal{G}$ has a continuous Haar system and is measurewise amenable, then $C^*(\mathcal{G}) \simeq C_r^*(\mathcal{G})$ [7]. The converse isn't known.

8. DYNAMICAL SYSTEMS AND CROSSED PRODUCTS

- **topological groups**: A topological group G acts on a C^*-algebra A by automorphisms via a group homomorphism $\alpha : G \to Aut(A)$ such that maps $x \mapsto \alpha_x(a)$ are continuous, for each $a \in A$. Then

$$L^1(G, A) = \{f : G \to A : f \text{ Borel measurable}, \int_G \|f(x)\| d\lambda(x) < \infty\}$$

is a Banach $*$-algebra with respect to the following operations and above norm

$$f * g(x) = \int_G f(y)\alpha_y(g(y^{-1}x))d\lambda(y), \; f^*(x) = \Delta(x^{-1})\alpha_x(f(x^{-1}))^*.$$

If π is a non-degenerate representation of A and u is a unitary representations of G (on the same Hilbert space $\mathcal{H}$) with

$$\pi(\alpha_x(a)) = u_x \pi(a) u_x^* \quad (a \in A, x \in G),$$

then $\{\pi, u, \mathcal{H}\}$ is called a covariant representation of the dynamical system (A, G, α). In this case the Bochner integral

$$(\pi \times u)(f) = \int_G \pi(f(x)) u_x d\lambda(x)$$

defines a non-degenerate $*$-representation of $L^1(G, A)$ and these exhaust all such representations. Also if

$$\tilde{\pi}(a)\xi(x) = \pi(\alpha_{x^{-1}}(a))\xi(x), \; \pi_L(x)\xi(y) = \xi(x^{-1}y),$$

for $x, y \in G, a \in A, \xi \in L^2(G, \mathcal{H})$, then $Ind(\pi) = \tilde{\pi} \times \pi_L$ is the left regular representation induced by π and we have the C^*-norms

$$\|f\| = \sup_{\pi, u} \|(\pi \times u)(f)\|, \; \|f\|_r = \sup_{\pi} \|Ind(\pi)(f)\|.$$

The completions of $L^1(G, A)$ with respect to these C^*-norms are called the full and reduced C^*-crossed products of A by G and are denoted by $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$, respectively. If $\alpha = tr$ is the trivial action we get

$$A \rtimes_\alpha G \simeq A \otimes_{max} C^*(G), \quad A \rtimes_{\alpha,r} G \simeq A \otimes_{min} C_r^*(G).$$

- **inverse semigroups**: A unital inverse semigroup S acts on a C^*-algebra A by partial automorphisms via a semigroup homomorphism $\beta : S \to PAut(A)$ sending s to the triple (β_s, E_{s^*}, E_s), where $\beta_s : E_{s^*} \to E_s$ is an isomorphism between two closed ideals of A (that's what a partial automorphism is). Then

$$L = \{f \in \ell^1(S, A) : f(s) \in E_s\}$$

is a Banach $*$-algebra with respect to the following operations and ℓ^1-norm

$$f * g(s) = \sum_{rt=s} \beta_r(\beta_{r^*}(f(r))g(t)), \ f^*(s) = \beta_s(f(s^*)^*).$$

If π is a non-degenerate representation of A and u is a representations of S (on the same Hilbert space $\mathcal{H}$) with

$$\pi(\beta_s(a)) = u_s\pi(a)u_{s^*} \quad (s \in S, a \in E_{s^*}),$$

then $\{\pi, u, \mathcal{H}\}$ is called a covariant representation of the dynamical system (A, S, β). In this case the WOT-convergent sum

$$(\pi \times u)(f) = \sum_{s\in S} \pi(f(s))u_s$$

defines a non-degenerate $*$-representation of L and these exhaust all (coherent) representations. Also we have the C^*-norm

$$\|f\| = \sup_{\pi,u} \|(\pi \times u)(f)\|,$$

and if $I = \{f \in L : \|f\| = 0\}$, the completions of L/I with respect to this C^*-norm is called the full C^*-crossed product of A by S and is denoted by $A \rtimes_\beta S$. If $\beta = tr$ is the trivial action and $A = \mathbb{C}$ we get $\mathbb{C} \rtimes_{tr} S \simeq C^*(G(S))$, where $G(S)$ is the maximal group homomorphic image of S. Also S naturally acts on $C^*(E_S)$ by

$$\beta_s(\delta_e) = \delta_{ses^*} \quad (s \in S, e \in E_S),$$

and $C^*(E_S) \rtimes_\beta S \simeq C^*(S)$ [16].

- **topological hypergroups**: There is no notion of hypergroup C^*-dynamical system, but one can easily define them for measured hypergroups.

- **topological groupoids**: A topological groupoid $\mathcal{G}$ acts on a C^*-bundle $\mathcal{A} = \{A_u\}_{u\in\mathcal{G}^{(0)}}$ by automorphisms via a map $\alpha : \mathcal{G} \to Aut(\mathcal{A})$ such that maps $(a, x) \mapsto \alpha_x(a)$ (with $a \in A_{s(x)}$) are continuous and maps $\alpha_x : A_{s(x)} \to A_{r(x)}$ are C^*-algebra isomorphisms, and $\alpha_{xy} = \alpha_x\alpha_y$, whenever $(x, y) \in \mathcal{G}^{(2)}$. $\mathcal{A}$ is assumed to be continuous [18] or upper-semicontinuous [16]. Pulling back via $r : \mathcal{G} \to \mathcal{G}^{(0)}$, we get the Banach bundle

$$r^*\mathcal{A} = \{(a, x) : a \in A_{r(x)}\}$$

with set of compactly supported continuous sections $C_c(\mathcal{G}, r^*\mathcal{A})$ endowed with the following operations

$$f * g(x) = \int_{\mathcal{G}^{r(x)}} f(y)\alpha_y(g(y^{-1}x))d\lambda^{r(x)}(y), \quad f^*(x) = \alpha_x(f(x^{-1}))^*.$$

Put $\|f\| = \sup_\pi \|\pi(f)\|$, where π ranges over all $*$-representations of the normed algebra $C_c(\mathcal{G}, r^*\mathcal{A})$ which are continuous from the inductive limit topology to WOT (automatic when $\mathcal{G}$ is r-discrete.) The completion of $C_c(\mathcal{G}, r^*\mathcal{A})$ in this C^*-norm is called the groupoid crossed product and is denoted by $\mathcal{A} \rtimes_\alpha \mathcal{G}$. The reduced C^*-crossed product $\mathcal{A} \rtimes_{\alpha,r} \mathcal{G}$ could be defined similar to the group case. The regular representations used here are exactly the representations induced from $C_0(\mathcal{G}^{(0)}, \mathcal{A})$ via the generalized conditional expectation $P : C_c(\mathcal{G}, r^*\mathcal{A}) \to C_c(\mathcal{G}^{(0)}, \mathcal{A})$ defined by restriction [7].

- If $\mathcal{G}$ has a continuous Haar system and is measurewise amenable, then $\mathcal{A} \rtimes_\alpha \mathcal{G} \simeq \mathcal{A} \rtimes_{\alpha,r} \mathcal{G}$ [7]. The converse isn't known.

9. TRANSITIONS

- **from groupoids to inverse semigroups**: For a topological groupoid $\mathcal{G}$, let's consider the set $\mathcal{G}^{op}$ consisting of those open Hausdorff $U \subseteq \mathcal{G}$ on which r and s are homeomorphisms with open ranges. If $\mathcal{G}$ is Hausdorrf and r-discrete (i.e. s, r are local homeomorphisms) $\mathcal{G}^{op}$ consists of open $\mathcal{G}$-sets and it is an inverse semigroup under the operations

$$vw = \{xy : (x, y) \in (v \times w) \cap \mathcal{G}^{(2)}\}, \quad v^* = \{x^{-1} : x \in v\}.$$

A subsemigroup S of $\mathcal{G}^{op}$ is full if it is a base for the topology of $\mathcal{G}$ and E_S is upward directed. In this case, if $\mathcal{G}$ acts by α on a C^*-bundle $\mathcal{A} = \{A_u\}_{u \in \mathcal{G}^{(0)}}$, then S acts on $C_0(\mathcal{G}, \mathcal{A})$ by

$$\beta_s(f)(u) = \begin{cases} \alpha_{us}(f(s^*us)) & u \in r(s) \\ 0 & \text{otherwise} \end{cases} \qquad (s \in S, f \in B_{s(s)}, u \in \mathcal{G}^{(0)}),$$

where

$$B_e = \{f \in C_0(\mathcal{G}, \mathcal{A}) : f = 0 \text{ off } e\} \quad (e \in E_S).$$

Conversely, if S acts by β on a $C_0(\mathcal{G}^{(0)})$-algebra A (a C^*-algebra with a faithful non-degenerate homomorphism ϕ from $C_0(\mathcal{G}^{(0)}$ into the algebra $ZM(A)$ of central multipliers of A,) then for

$$I_u = \{f \in C_0(\mathcal{G}^{(0)}) : f(u) = 0\} \quad (u \in \mathcal{G}^{(0)})$$

and $A_u = A/(\phi(I_u)A)$, $\mathcal{G}$ acts on $\mathcal{A} = \bigcup_{u \in \mathcal{G}^{(0)}} A_u$ by

$$\alpha_x(f(s(x))) = \beta_s(f)(r(x)) \quad (x \in s \in S, f \in B_{s(s)}).$$

In this case we have $A \rtimes_\beta S \simeq \mathcal{A} \rtimes_\alpha \mathcal{G}$. In particular, when $\alpha = tr$, $S = \mathcal{G}^{op}$, and $\mathcal{A} = C_0(\mathcal{G}^{(0)})$,

$$C^*(\mathcal{G}) \simeq C_0(\mathcal{G}^{(0)}) \rtimes_\beta \mathcal{G}^{op} \, [16].$$

- **from inverse semigroups to groupoids**: If S is an inverse semigroup with idempotents $E = E_S$, a filter in E is a subsemigroup $A \leq E$ such that $e \in A, f \in E$ and $f \geq e$ implies $f \in A$. Take $X = \ell^1(E)^\curlyvee$ (the maximal ideal space of the commutative Banach algebra $\ell^1(E)$ which could be identified with the space of nonzero characters with w^*-topology,) then for each $x \in X$, $A_x = \{e \in E : x(e) = 1\}$ is a filter in E and these exhaust all possible filters (given a filter A, simply put $x = \chi_A$ to get $A = A_x$.) In particular for $e \in E$, $\bar{e} = \{f \in E : f \geq e\}$ is a filter in E and $e \mapsto \bar{e}$ maps E onto a dense subset of X. For $e \in E$ and $s \in S$ put

$$D_e = \{x \in X : x(e) = 1\}, \quad D_s = D_{ss^*}, \quad R_s = D_{s^*}.$$

Then S acts on X by

$$x.s(e) = x(ses^*) \quad (s \in S, x \in X, e \in E)$$

and the map $x \mapsto x.s =: \beta(s)(x)$ is a homeomorphism from D_s onto R_s. Put

$$\Sigma = \{(x, s) : x \in D_s, s \in S\}$$

and set $(x, s) \sim (y, t)$ iff $x = y$, $es = st$ and $x(e) = 1$, for some $e \in E$. Then $\mathcal{G}_S = \Sigma / \sim$ is a groupoid under operations

$$[x, s].[x.s, t] = [x, st], \quad [x, s]^{-1} = [x.s, s^*] \quad (s, t \in S, x \in X),$$

where $[x, s]$ is the equivalent class of (x, s). This is called the universal groupoid of S. It is an r-discrete (Hausdorff, when S is E-unitary) groupoid with $\mathcal{G}_S^{(0)} = X$ and $\mathcal{G}_S^{op} = \{[U, s] : s \in S, U \subseteq D_s \text{ open}\}$ with $S \leq \mathcal{G}_S^{op}$ full.

If S acts on a C^*-algebra A by subquotients [12] through α, then $E \rtimes_\alpha A$ is a $C_0(\mathcal{G}_S^{(0)})$-algebra and $\mathcal{G}_S$ acts on $E \rtimes_\alpha A$ by $\tilde{\alpha}$ such that

$$\mathcal{G}_S \rtimes_{\tilde{\alpha}} (E \rtimes_\alpha A) \simeq S \rtimes_\alpha A, \quad \mathcal{G}_S \rtimes_{\tilde{\alpha}, r} (E \rtimes_\alpha A) \simeq S \rtimes_{\alpha, r} A.$$

In particular

$$C^*(\mathcal{G}_S) \simeq C^*(S), \quad C_r^*(\mathcal{G}_S) \simeq C_r^*(S) \, [12], [14].$$

10. OPEN PROBLEMS

There are non-compact groups (Moore groups) whose irreducible representations are all finite dimensional.

- *Is there a Moore theory for inverse semigroups or topological hypergroups?*

If S is an inverse semigroup, then $C_r^*(S)$ and $VN(S)$ are the C^*-algebra and von-Neumann algebra generate by $\{\pi_L(s) : s \in S\}$ in $\mathcal{B}(\ell^2(S))$, respectively. The necessary and sufficient condition for nuclearity of $C_r^*(S)$ are obtained in [4]. Also in [14], it is shown that if all maximal subgroups of S (indexed by E_S) are amenable, then $VN(S)$ is injective.

- *Characterize all inverse semigroups S for which $VN(S)$ is injective.*

One possible attack is to use the characterization of amenability of $\mathcal{G}_S$ given in [4] and relating injectivity of $VN(S)$ to that of $VN(\mathcal{G}_S)$.

- *For which topological groupoids $\mathcal{G}$, $C^*(\mathcal{G}) \simeq C^*_r(\mathcal{G})$? What about $\mathcal{G} \rtimes_\alpha \mathcal{A} \simeq \mathcal{G} \rtimes_{\alpha,r} \mathcal{A}$?*

A sufficient condition is that $\mathcal{G}$ has a continuous Haar system and it is measurewise amenable [7].

- *For which inverse semigroups S, $C^*(S) \simeq C^*_r(S)$? What about $S \rtimes_\alpha A \simeq S \rtimes_{\alpha,r} A$?*

A sufficient condition is given in [4].

There is a well-established theory of C^*-bundles over (discrete) groups [10] and their associated full and reduced C^*-algebras (which are natural generalizations of C^*-crossed products.)

- *What about C^*-bundles over inverse semigroups?*

There are some attacks, but many things left to be explored.

For a topological group G, there is a rich operator space structure on the Banach algebra $B(G) = C^*(G)^*$ and its closed ideals including the well known Fourier algebra $A(G)$. In particular $A(G)$ is operator amenable iff G is amenable [20].

- *What about inverse semigroups or groupoids?*

The Fourier algebras of inverse semigroups and topological groupoids are studied in [5],[6],[19].

Green has shown that the Mackey-Rieffel machine could be applied to group C^*-crossed products in order to study induced representations from subgroups.

- *Is there a corresponding technique to study induced representations of groupoid or inverse semigroup crossed products?*

For groupoids we have the notion of imprimitivity groupoid by Muhly and Williams. There is a work in progress for induced representations of groupoid crossed products [1].

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