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INTRODUCTION TO GKM THEORY

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ABSTRACT. GKM theory has been developed to compute torus equivariant cohomologies on some nice space called GKM space. A set of GKM spaces consists of many classes in the spaces with torus actions, e.g., the torus manifold is the GKM space. So we can expect that to study GKM spaces will lead us in a deep understanding of torus actions. The aim of this article is to introduce GKM theory.

1. INTRODUCTION

Let G be a Lie group and M be a smooth compact manifold. We call a smooth map $\varphi : G \times M \to M$ a *G-action on M*, if φ satisfies the following two properties:

(1) $\varphi(e, x) = x$ for the identity element $e \in G$ and $x \in M$;
(2) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for $g, h \in G$ and $x \in M$.

On the other hand, if there exists such φ then we call that *G acts on M* or *M is a G-space*. In particular, in this article, we put $G = T^n$, i.e., the n-dimensional, compact commutative group (we call it the *n-dimensional torus*).

GKM theory is to study an *equivariant cohomology* (see Section 2) of T^n-action on even dimensional smooth manifold M^{2m} with finite *fixed points*, i.e., elements $x \in M$ such that $\varphi(g, x) = x$ for all $g \in T^n$. In the paper [GKM98], Goresky, Kottwitz and MacPherson showed that an equivariant cohomology ring of such torus action with suitable condition (we call it a *GKM manifold*) has a combinatorial description, that is, we can describe the equivariant cohomology ring structure of GKM manifolds by using the combinatorial notion (we call it a *GKM description*). From their point of view, in order to define the equivariant cohomology of GKM

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manifolds, we do not need to use a chain complex any more; instead, we may only use graphs with *axial functions* induced by torus actions. For example, in the paper [KT03], Knutson and Tao adopted the GKM description as the definition of the equivariant cohomology on complex Grassmannians. Moreover, we can expect its applications to graph theory from GKM theory (e.g. see [GZ01]). In topological point of view, the most interesting part of this theory is to expect to make bridges among several theories which developed individually, e.g., the theory of torus manifolds or the equivariant Schubert calculus and etc.

The aim of this article is an introduction of GKM theory. The organization of this article is as follows. In Section 2, we recall some basic facts on the equivariant cohomology. In Section 3, we define the GKM manifold and introduce the GKM description of the equivariant cohomology of GKM manifolds. In Section 4, we abstractly define the GKM graph and its graph equivariant cohomology. In Section 5, as a test case, we compute the ring structure of some graph equivariant cohomology of GKM graph. In Section 6, finally we give the principal (but it is supposed to be open-ended) problem of GKM theory.

2. Basics of equivariant cohomology

In this section, we overview the basics of equivariant cohomology. Basic references are [H75][1] or [GS99][2] for equivariant cohomology, [B72] or [Ka91] for compact group actions and [MT91] for topology of compact Lie groups.

2.1. Basics of group actions.

We first recall the basic terminologies of group actions. Let $\varphi : G \times M \to M$ be a G-action on M and $x \in M$. We often denote it as $G \curvearrowright^{\varphi} M$. In this article, we always consider group actions under the smooth category. So we can define the following representation from φ:

$$\Phi : G \to \mathrm{Diff}(M)$$

by

$$\Phi(g) = \varphi(g, \ ),$$

where the symbol $\mathrm{Diff}(M)$ represents the set of all diffeomorphisms on M and it has the compact-open topology. If Φ is injective, i.e., $\ker \Phi = \{e\}$ then we call this action φ is an *effective action*.

In our notation, the symbol $G(x)$ represents the *G-orbit of x* and G_x represents the *isotropy subgroup* of G on x, i.e.,

$$G(x) = \{\varphi(g, x) \in M \mid g \in G\}$$

[1]The book [H75] is written from algebraic topological point of view.

[2]The book [GS99] is written from differential topological point of view related with GKM theory.

and

$$G_x = \{g \in G \mid \varphi(g, x) = x\},$$

respectively. If G is compact, $G(x)$ is diffeomorphic to G/G_x and $\ker \Phi = \cap_{x \in M} G_x$. The symbol M/G denotes the *orbit space of G-action on M*, i.e, the set of all orbits.

By using the Milnor construction of G, we can construct the space EG which satisfies the following two conditions:

(1) EG is contractible;
(2) G acts on EG *freely*, i.e., $G_x = \{e\}$ for all $x \in EG$.

Because G acts on EG, we have its orbit space EG/G. The symbol BG represents EG/G and we call it a *classifying space of G*.

2.2. **Equivariant cohomology.** Next we define the equivariant cohomology and recall its basic facts. Let M be a G-space and φ a G-action on M. Now the product space $EG \times M$ has a diagonal G-action by

$$(a, x) \mapsto (ag^{-1}, \varphi(g, x))$$

where $a \in EG$, $x \in M$ and $g \in G$. Therefore, we can take its orbit space $(EG \times M)/G = EG \times_G M$. We call $EG \times_G M$ the *Borel construction* (or *homotopy quotient*). The symbol M_G also represents the Borel construction of G-space M. Now we may define an *equivariant cohomology* of G-space M.

Definition 2.1 (Equivariant cohomology)**.** Let M be a G-space. The *equivariant cohomology* of M is the ordinary cohomology of M_G. We describe the equivariant cohomology as $H^*_G(M)$, i.e.,

$$H^*_G(M) = H^*(M_G)(= H^*(EG \times_G M))$$

Remark 2.2. In this paper, we consider the integer coefficient (i.e., $\mathbb{Z}$-coefficient) or the rational coefficient (i.e., $\mathbb{Q}$-coefficient) as the coefficient of cohomology rings. The symbol $H^*(X)$ represents the cohomology with $\mathbb{Z}$-coefficient and the symbol $H^*(X; \mathbb{Q})$ represents the cohomology with $\mathbb{Q}$-coefficient.

Because G acts on EG freely, we have the following fibration for the Borel construction of M:

$$M \xrightarrow{\iota} EG \times_G M \xrightarrow{\pi} BG.$$

Therefore, we also have the following sequence for the equivariant cohomology of M:

$$H^*(M) \xleftarrow{\iota^*} H^*_G(M) \xleftarrow{\pi^*} H^*(BG). \tag{2.1}$$

Hence, we see that $H^*_G(M)$ is not only ring but also $H^*(BG)$-algebra via the representation π^*. In general, the ring structure of $H^*(BG)$ is still open problem[3]. However, if $G = T^n$, the ring structure of $H^*(BT)$ is known as the polynomial ring, i.e.,

$$H^*(BT^n) \simeq \mathbb{Z}[x_1, \ldots, x_n]$$

where $\deg x_i = 2$ for $i = 1, \ldots, n$.

2.3. **Computations of equivariant cohomology for easy cases.** In the final part of this section, we compute $H^*_G(M)$ for special cases.

2.3.1. *The case M is one point.* Assume $M = \{*\}$, i.e., one point. Then we have $EG \times_G \{*\} \cong BG$. Therefore, $H^*_G(\{*\}) \simeq H^*(BG)$.

2.3.2. *The case G-action on M is freely.* Assume G acts on M freely. Then we have the following fibration for $EG \times_G M$:

$$EG \longrightarrow EG \times_G M \longrightarrow M/G.$$

Because EG is contractible, we have that $EG \times_G M \to M/G$ induces the homotopy equivariant map. Therefore, $H^*_G(M) \simeq H^*(M/G)$. In particular, for all subgroups $K \subset G$, we have $H^*_K(G) \simeq H^*(G/K)$.

2.3.3. *The case when G-action on M is transitive.* Assume G acts on M *transitively*, i.e., $G(x) = M$ for $x \in M$. In this case, we have $M \cong G/H$ for some subgroup $H \subset G$. Now we have the following equation:

$$EG \times_G (G/H) \cong EG/H \cong_h EH/H = BH,$$

where $\cong_h$ represents homotopy equivalent. Therefore, $H^*_G(G/H) \cong H^*(BH)$.

2.3.4. *Summary and problem.* Summarizing the above arguments, we have the following proposition.

Proposition 2.3. *We have the following relations for $H^*_G(M)$:*

(1) *if $M = \{*\}$, then $H^*_G(*) \simeq H^*(BG)$;*
(2) *if G acts on M freely, then $H^*_G(M) \simeq H^*(M/G)$;*
(3) *if G acts on M transitively and $M \cong G/H$ for some subgroup H, then $H^*_G(G/H) \simeq H^*(BH)$.*

In general, group actions on M are more complicated than the above cases. So we can naturally ask the following question:

Problem 2.4. Find a useful method to compute $H^*_G(M)$ for general cases.

[3]This is for $\mathbb{Z}$-coefficient. For $\mathbb{Q}$-coefficient (rational coefficient), the ring structure of $H^*(BG; \mathbb{Q})$ is known for all compact Lie groups (see [MT91, Chapter 6]).

One of the answers to Problem 2.4 is *GKM theory*. We will argue it from the next section.

3. GKM manifold and its equivariant cohomology

Henceforth, we put $G = T^n$, i.e., $T^n = S^1 \times \cdots \times S^1$ (n times product of S^1's) and we assume that M^{2m} is a $2m$-dimensional, connected, finite T-manifold; therefore, it has the finite T-CW complex structure[4], where $n \leq m$. Before we define a GKM manifold, we introduce the notion of an *equivariantly formal space*.

3.1. **Equivariantly formal space.**

Recall the sequence (2.1):

$$H^*(BT) \xrightarrow{\pi^*} H_T^*(M) \xrightarrow{\iota^*} H^*(M).$$

Now we may define an equivariantly formal space.

Definition 3.1 (Equivariantly formal space)**.** If $\iota^* : H_T^*(M) \to H^*(M)$ is surjective, then we call M an *equivariantly formal space*.

Remark 3.2. In the paper [GKM98], if $\iota^* : H_T^*(M; \mathbb{Q}) \to H^*(M; \mathbb{Q})$ is surjective then we call M an equivariantly formal space. In this case, this property is equivalent to $H_T^*(M; \mathbb{Q})$ is the free $H^*(BT; \mathbb{Q})$-module. However, for the $\mathbb{Z}$-coefficients, this is not true in general (see [FP06, Example 5.2]).

We also remark the following proposition. This proposition is one of the criterion when $H_T^*(M)$ is the free $H^*(BT)$-module for the equivariantly formal space.

Proposition 3.3. *Let M a simply connected, equivariantly formal T-space. Then we also have the following properties:*

1. $\pi^* : H^*(BT) \to H_T^*(M)$ *is injective;*
2. $H_T^*(M)$ *is free $H^*(BT)$-module.*

Proof. Due to [MT91, Chapter 3: Lemma 2.15, Theorem 4.4], the Serre spectral sequence of the fibration $M \to M_T \to BT$ collapses. Therefore, by using [MT91, Chapter 3: Theorem 4.2], we have these properties. □

Before we state the key theorem in GKM theory, we introduce the notion *k-skelton*. Let M be a T-space. Then we have the following sequence constructed by T-invariant subspaces:

$$M^T = M_0 \subset M_1 \subset \cdots \subset M_n = M^{2m},$$

where

$$M_k = \{x \in M \mid \dim T(x) \leq k\}$$

[4]This assumption is more general than the compactness. Moreover, from this assumption, the cohomology ring of M is finite dimension in each degree, i.e., $\dim H^i(M) < \infty$ for all $i = 0, \ldots, 2m$.

called the *k-skelton* of T-action on M. Note that $M_0 = M^T$, i.e., the set of fixed points and $M_n = M^{2m}$. Now we may state the following important key theorem in GKM theory (see [FP07, Theorem 2.3]):

Theorem 3.4 (ABFP exact sequence)**.** *Let M be an equivariantly formal space. If the isotropy subgroup T_x is connected for all $x \in M$, then the following sequence is exact:*

$$\begin{aligned} & 0 \longrightarrow H_T^*(M) \xrightarrow{\rho^*} H_T^*(M^T) \xrightarrow{\partial_0^*} H_T^{*+1}(M_1, M_0) \\ \xrightarrow{\partial_1^*} \quad & H_T^{*+2}(M_2, M_1) \xrightarrow{\partial_2^*} \cdots \xrightarrow{\partial_{n-2}^*} H_T^{*+n-1}(M_{n-1}, M_{n-2}) \\ \xrightarrow{\partial_{n-1}^*} \quad & H_T^{*+n}(M_n, M_{n-1}) \longrightarrow 0, \end{aligned}$$

where the map $\rho : M^T \to M$ represents the embedding and $\partial_\ell^* : H_T^*(M_\ell, M_{\ell-1}) \to H_T^{*+1}(M_{\ell+1}, M_\ell)$ is defined by the long sequence of $(M_{\ell+1}, M_\ell, M_{\ell-1})$.

We call the sequence in Theorem 3.4 the *Atiyah-Bredon-Franz-Puppe sequence* (the *ABFP sequence* for short).

If we put some assumption on M then we can change the assumption for the T-action in Theorem 3.4 to weaker condition as follows.

Theorem 3.5. *Let M be a simply connected, equivariantly formal space. If $T_x/T_x^o \simeq \mathbb{Z}_{\ell(x)}$ or $\{e\}$ for all $x \in M$ and some $\ell(x) \in \mathbb{Z} - \{0, \pm 1\}$ (where T_x^o is the identity component of T_x), then the ABFP sequence is exact.*

Proof. Using Proposition 3.3 and [FP07, Theorem 2.2], we have this theorem. □

As a corollary of the exactness of the ABFP sequence, we have the following famous facts:

Corollary 3.6. *Let $\rho_1 : M^T \to M_1$ be the natural embedding. If the ABFP sequence is exact, then we have the following diagram:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_T^*(M) & \xrightarrow{\rho^*} & H_T^*(M^T) & \xrightarrow{\partial_0^*} & H_T^{*+1}(M_1, M^T) \\ & & & & \uparrow \rho_1^* & & \\ & & & & H_T^*(M_1) & & \end{array}$$

such that

- *ρ^* is injective (the localization theorem);*
- *$\mathrm{Im}\ \rho^* = \ker \partial_0^* = \mathrm{Im}\ \rho_1^*$ (the Chang-Skjelbred lemma).*

Therefore, in order to compute $H_T^*(M)$ of the equivariantly formal T-space with suitable conditions for T_x, we may only compute the image of ρ_1^*, i.e.,

$$\mathrm{Im}\ \rho_1^* = \rho_1^*(H_T^*(M_1)) \subset H_T^*(M^T).$$

This fact is one of the motivations for the definition of GKM manifold.

Remark 3.7. In [FP07, Theorem 2.4], there are some sufficient conditions which induce the *Chang-Skjelbred sequence*, i.e., the first sequence in Corollary 3.6, is exact.

For the $\mathbb{Q}$-coefficients, we do not need to assume T_x is connected in Theorem 3.4.

3.2. **GKM manifold.** In this section, we define a *GKM manifold*. Before we define it, we recall some notations. Let $p \in M^T$ and $\dim M = 2m$. Because we assume the T-action on M is smooth in this paper, we can induce a T-representation of the tangent space T_pM at $p \in M^T$ from $T \curvearrowright M$. We call this representation a *tangential representation*. Since T is isomorphic to $S^1 \times \cdots \times S^1$, the tangent space T_pM can be decomposed into the irreducible T-representations as follows:

$$T_pM \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_m),$$

where $\alpha_i \in \mathrm{Hom}(T^n, S^1)$ and $V(\alpha_i) \simeq \mathbb{C}$ (for all $i = 1, \ldots, m$).

Now we may define a GKM manifold.

Definition 3.8 (GKM manifold)**.** Let M^{2m} be a $2m$-dimensional (simply connected) manifold with effective T^n-action, where $m \geq n$. The manifold M^{2m} is called the *GKM manifold* if it satisfies the following three conditions:

(1) M is equivariantly formal (and $T_x/T_x^o \simeq \mathbb{Z}_{\ell(x)}$ for all $x \in M$);
(2) M^T is finite;
(3) For $T_pM \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_m)$, the set $\{\alpha_1, \ldots, \alpha_m\} \subset \mathrm{Hom}(T^n, S^1) \simeq H^2(BT^n)$ is pairwise linearly independent, i.e., for all pair (α_i, α_j)'s $(i \neq j)$ are linearly independent in $H^2(BT^n)$.

Here, we give the meaning of each assumptions in Definition 3.8. The first assumption is for using the Chang-Skjelbred exact sequence in Corollary 3.6. The second assumption is for the finiteness of $\dim_{H^*(BT)} H_T^*(M^T)$. Therefore, we have

$$H_T^*(M^T) \simeq \oplus_{p \in M^T} H_T^*(p) \simeq \oplus_{p \in M^T} \mathbb{Z}[x_1, \ldots, x_n].$$

The third assumption is equivalent to the following geometric condition:

$$\dim M_1 = 2,$$

where M_1 is the 1-skelton of $T \curvearrowright M$. This condition is also identified with the following equation:

$$M_1/T = \Gamma,$$

where Γ is the graph (also see Section 4.1) such that the set of vertices $\mathcal{V}^\Gamma = M^T$ and the set of edges $\mathcal{E}^\Gamma$ (resp. legs $\mathcal{L}^\Gamma$) is the set of invariant spheres which connect two fixed points on M (resp. invariant $\mathbb{C}$ whose origin is a fixed point). Note that if M is compact then there is no legs in the graph M_1/T, and the second assumption holds automatically.

Example 3.9. For example, the following spaces are GKM manifolds:

(1) Equivariantly formal torus manifolds (i.e., torus manifolds with $H^{odd}(M) = 0$):
(2) Hypertoric manifolds with residual S^1-actions:
(3) Homogeneous spaces G/H with same rank, i,e, $T \subset H \subset G$ for maximal torus $T \subset G$ and H and G are connected:

Proof. For the first example, see the papers [MP06] and [MMP07]. For the second, see the papers [HP04] and [HH05]. For the third, see the paper [GHZ06]. □

The location of torus manifolds in GKM manifolds are as the following Figure 1. In Figure 1, each parenthesis represents the corresponding combinatorial theory.

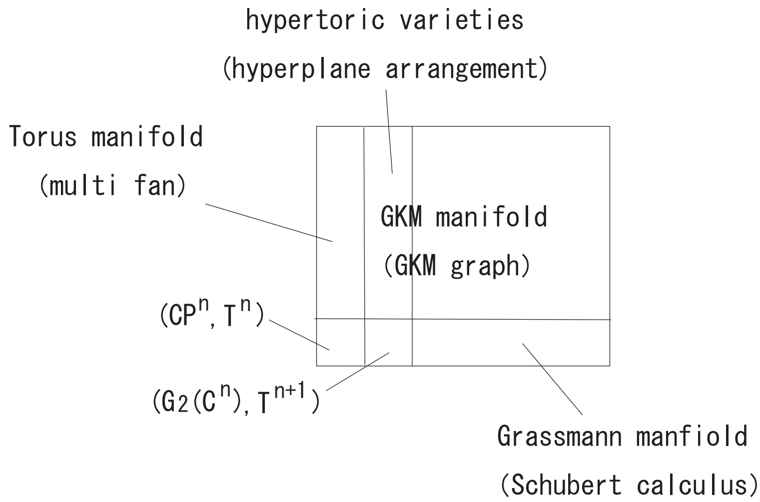

FIGURE 1. A set of GKM manifolds.

Therefore, in some sense, we can regard a *GKM graph* as the universal combinatorial theory which describes GKM manifolds (see Section 4).

We give more concrete examples in Example 3.9 (1), where a *torus manifold* is a $2n$-dimensional T^n-manifold with fixed points:

Example 3.10. Let $S^2 \subset \mathbb{R} \oplus \mathbb{C}$ be a 2-dimensional sphere. Figure 2 is the picture of $T^1 \curvearrowright S^2$.

Precisely, this is defined as follows. The 1-dimensional torus T^1 acts on S^2 by

$$\varphi(t, (r, z)) = (r, tz),$$

where $(r, z) \in S^2$, i.e., $r \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $r^2 + |z|^2 = 1$. Then this is an equivariantly formal torus manifolds because this action has 2 fixed points and $H^{odd}(S^2) = 0$. Therefore, this is one of the examples of GKM manifolds.

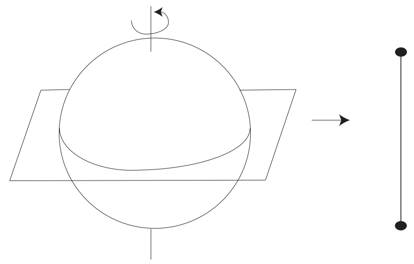

FIGURE 2. $T^1 \curvearrowright S^2$ and S^2/T^1.

Example 3.11. Let $\mathbb{C}P(n) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ be a n-dimensional complex projective space. Then the n-dimensional torus T^n acts on $\mathbb{C}P(n)$ as follows:

$$\varphi((t_1, \ldots, t_n), [z_0 : z_1 : \cdots : z_n]) = [z_0 : t_1 z_1 : \cdots : t_n z_n],$$

where $[z_0 : z_1 : \cdots : z_n]$. Figure 3 is the 1-skelton of $T^2 \curvearrowright \mathbb{C}P(2)$.

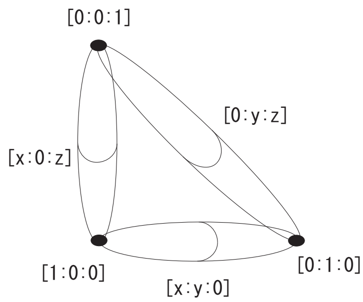

FIGURE 3. The 1-skelton of $T^2 \curvearrowright \mathbb{C}P(2)$.

We can easily show that this is one of the examples of GKM manifolds because the complex projective space with the above action is the equivariantly formal torus manifolds.

We can compute the equivariant cohomology of GKM manifolds by using the following theorem.

Theorem 3.12 (GKM description)**.** *Let M be a GKM manifold. Then we have the following combinatorial description for the equivariant cohomology:*

$$H_T^*(M) \simeq \{f : M^T \to H^*(BT) \mid f(p) - f(q) \equiv 0 \pmod{\alpha(pq)}\},$$

where $p, q \in M^T$ are connected by an edge in M_1/T and $\alpha(pq) \in H^2(BT)$ is the T-representation along the tangent space of S^2 which connects p and q.

Before we explain an outline of the proof of this theorem, we will draw the picture of the T-representation along the tangent space of S^2 (see Figure 4).

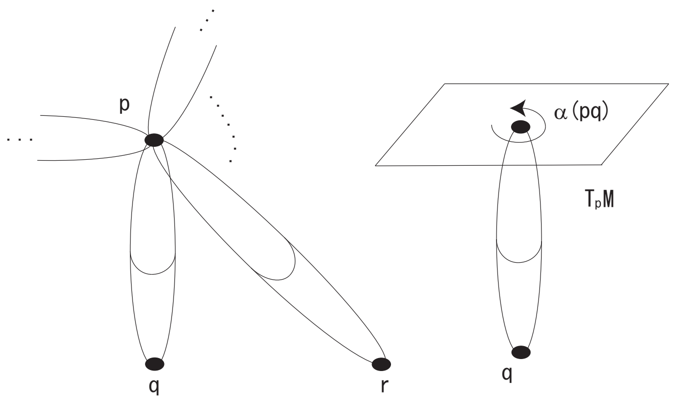

FIGURE 4. This is the picture of decomposition of the tangential representation on the fixed point $p \in M^T$.

The left figure in Figure 4 represents the 1-skeltons around fixed point $p \in M^T$, and in the right figure we pick up one of the 1-skeltons around p. By the definition (3) of the GKM manifold, we have the following decomposition on $p \in M^T$:

$$T_pM \simeq V(\alpha_1) \oplus \cdots \oplus V(\alpha_m).$$

Then there are invariant 2-spheres S_i^2 $(i = 1, \ldots, m)$ such that $p \in (S_i^2)^T$ and its tangential representation is equal to α_i. In the figure, one of the decomposition of T_pM represents by the right figure as the representation of the T-action on T_pS^2, where S^2 is the invariant sphere which connects p and q. Moreover, we can get the real two dimensional representation from the T-action on T_pS^2 which corresponds to one of the factors in the decomposition of the tangential representation. We denote this representation as $\alpha(pq) : T \to S^1$ (see the right figure).

Outline of the proof. By using Corollary 3.6, we have

$$H_T^*(M) \simeq \iota_1^*(H_T^*(M_1)) \subset H_T^*(M^T).$$

Therefore, we may only need to know the ring structure of $\iota_1^*(H_T^*(M_1))$. This ring structure is easy to know by using the Mayer-Vietoris argument. □

From the next section, we will give the abstract description of the above facts by using the *GKM graph* and its *graph equivariant cohomology.*

4. GKM GRAPH AND GRAPH EQUIVARIANT COHOMOLOGY

In this section, we define the GKM graph and its graph equivariant cohomology.

Motivated by Theorem 3.12, we introduce the graph with *axial functions* and its cohomology ring called *graph equivariant cohomology ring.* These ideas are introduced by Guilleminn-Zara in the paper [GZ01] by abstracting from Hamiltonian torus actions, Maeda-Masuda-Panov in [MMP07] abstracting from equivariantly formal torus manifolds, Harada-Holm in [HH05] for hypertoric manifolds and Kuroki in [Ku] abstracting from the Harada-Holm's one.

4.1. **Graph.** First of all we recall the graph Γ. A *graph* Γ[5] is the set of finite vertices $\mathcal{V}^\Gamma$, the set of edges E^Γ and the set of legs L^Γ, where an *edge* is a one dimensional interval which connects two vertices pq and a *leg* is a one dimensional half line outgoing from one vertex p. The symbol $\mathcal{E}^\Gamma$ represents $E^\Gamma \cup L^\Gamma$. The Figure 5 is the picture of examples of graphs in this paper.

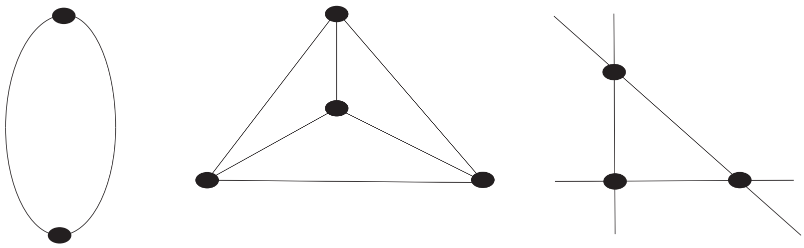

FIGURE 5. This is the picture of graphs. The left graph is a 2-valent graph with a multi-edge, the middle graph is a 3-valent graph, and the right graph is a 4-valent graph with 6 legs.

In this paper, we assume that if $pq \in E^\Gamma$ then $qp \in E^\Gamma$, i.e., we distinguish two orientations on one edge. We denote the initial vertex of $e \in \mathcal{E}^\Gamma$ as $i(e)$ and the terminal vertex as $t(e) \in \mathcal{E}^\Gamma$, e.g., $i(pq) = p$ and $t(pq) = q$ if $e = pq$ is the edge, and $i(e) = p$ and $t(e) = \emptyset$ if e is the leg. Moreover, we introduce the notation that $\mathcal{E}^\Gamma_p = \{e \in \mathcal{E}^\Gamma \mid i(e) = p\}$. In this paper, we assume $\#\mathcal{E}^\Gamma_p = m$ for all $p \in \mathcal{V}^\Gamma$ where $\#\mathcal{E}^\Gamma_p$ is the cardinality of the set $\mathcal{E}^\Gamma_p$, and we call such graph an *m-valent graph*.

4.2. **GKM graph.** Let $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$ be an m-valent graph. In order to define a GKM graph, we will define a *connection* and an *axial function*.

Before we define a connection, we prepare the set $\theta = \{\theta_{pq} \mid pq \in E^\Gamma\}$ which is a collection of bijective maps

$$\theta_{pq} : \mathcal{E}^\Gamma_p \to \mathcal{E}^\Gamma_q$$

for all edges $pq \in E^\Gamma$. Since Γ is an m-valent graph, we have $|\mathcal{E}^\Gamma_p| = m = |\mathcal{E}^\Gamma_q|$ for all $p,\ q \in \mathcal{V}^\Gamma$. Hence the bijective map θ_e always exists for all edges $e \in E^\Gamma$.

Definition 4.1. A *connection* on Γ is the set $\theta = \{\theta_{pq} |\ pq \in E^\Gamma\}$ which satisfies the following two conditions:

- $\theta_{qp} = \theta_{pq}^{-1}$;
- $\theta_{pq}(pq) = qp$.

[5]The definition of the graph in this paper is a part of the definition of the *hypergraph*.

Remark 4.2. We can easily show that an m-valent graph Γ admits different $((m-1)!)^g$ connections, where g is the number of (unoriented) edges E^Γ.

Next we define an axial function. In order to define it, we prepare some notations. Let $\mathfrak{t}$ be a Lie algebra of T, $\mathfrak{t}_\mathbb{Z}$ a lattice of $\mathfrak{t}$, and $\mathfrak{t}^*$ the dual algebra of $\mathfrak{t}$. The symbol $\mathrm{Hom}(T, S^1)$ represents a set of all homomorphisms from the group T to S^1, and we know that it can be regarded as a lattice of the dual algebra $\mathfrak{t}^*_\mathbb{Z}$. Hence, we have the identification $\mathrm{Hom}(T, S^1) \simeq \mathfrak{t}^*_\mathbb{Z} \simeq H^2(BT)$.

Definition 4.3. An *axial function*

$$\alpha : \mathcal{E}^\Gamma \to \mathrm{Hom}(T, S^1) \simeq \mathfrak{t}^*_\mathbb{Z} \simeq H^2(BT)$$

is a map which satisfies the following condition:

- $\alpha(qp) = \pm\alpha(pq)$ for all edges $pq \in E^\Gamma$.

Definition 4.4 (GKM graph)**.** Let $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ be a collection of an m-valent graph $\Gamma = (\mathcal{V}^\Gamma,\ \mathcal{E}^\Gamma)$, a connection θ on Γ, and an axial function

$$\alpha : \mathcal{E}^\Gamma \to \mathrm{Hom}(T,\ S^1) \simeq \mathfrak{t}^*_\mathbb{Z} \simeq H^2(BT).$$

We call $\mathcal{G} = (\Gamma,\ \alpha,\ \theta)$ a *GKM graph* if it satisfies the following conditions for all $p \in \mathcal{V}^\Gamma$:

(1) $\alpha(\mathcal{E}^\Gamma_p) = \{\alpha(e) \mid e \in \mathcal{E}^\Gamma_p\}$ is *pairwise linearly independent* for all $p \in \mathcal{V}^\Gamma$, that is, for two distinct elements $e_1,\ e_2 \in \mathcal{E}^\Gamma_p$, $\alpha(e_1),\ \alpha(e_2)$ are linearly independent in $\mathfrak{t}^*_\mathbb{Z}$;
(2) α satisfies the *congruence relation* for all edges $pq \in E^\Gamma$, that is, the relation $\alpha(e) - \alpha(\theta_{pq}(e)) \equiv 0 \pmod{\alpha(pq)}$ holds for all $e \in \mathcal{E}^\Gamma_p$.

Example 4.5. Figure 6 is an example of a GKM graph. Here, we attach the axial function $\alpha : \mathcal{E}^\Gamma \to Hom(T^2, S^1) \simeq H^2(BT) =<\alpha, \beta>$ as follows, where we regard $\alpha : T^2 \to S^1$ as the projection of the 1st coordinate and $\beta : T^2 \to S^1$ as the projection of the 2nd coordinate:

$$\begin{aligned} \alpha(pq) &= \alpha, & \alpha(qp) &= -\alpha; \\ \alpha(pr) &= \beta, & \alpha(rp) &= -\beta; \\ \alpha(qr) &= -\alpha + \beta, & \alpha(rq) &= \alpha - \beta. \end{aligned}$$

On the other hand, the GKM graph can be defined from the GKM manifold by taking the tangential representation as its axial function. For example, recall Example 3.11 for the $n = 2$ case (also see Figure 3). We may regard the tangent

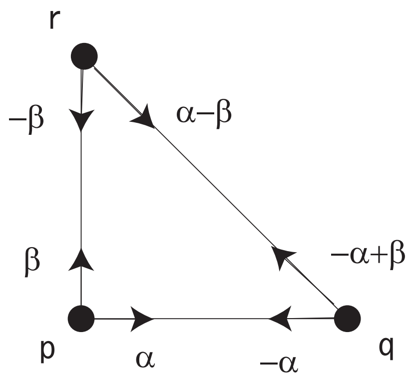

FIGURE 6. GKM graph. Here, α, β are the generators in $H^2_{T^2}(pt)$.

space of $p = [1 : 0 : 0]$, $q = [0 : 1 : 0]$, $r = [0 : 0 : 1]$ as

$$T_p\mathbb{C}P(2) = \{[1 : z_1 : z_2] \mid z_1, z_2 \in \mathbb{C}\} \simeq V(\alpha) \oplus V(\beta);$$
$$T_q\mathbb{C}P(2) = \{[z_0 : 1 : z_2] \mid z_0, z_2 \in \mathbb{C}\} \simeq V(-\alpha) \oplus V(\beta - \alpha);$$
$$T_r\mathbb{C}P(2) = \{[z_0 : z_1 : 1] \mid z_0, z_1 \in \mathbb{C}\} \simeq V(-\beta) \oplus V(\alpha - \beta).$$

Attaching these tangential representations along the invariants 2-spheres, we have the GKM graph as Figure 6.

If a restricted triple $(\Gamma', \ \alpha|_{\Gamma'}, \ \theta|_{\Gamma'}) \subset \mathcal{G}$ to a subgraph Γ' satisfies the definition of the GKM graph, then we call $\mathcal{G}' = (\Gamma', \ \alpha|_{\Gamma'}, \ \theta|_{\Gamma'})$ a *GKM subgraph* of $\mathcal{G}$.

4.3. **Graph equivariant cohomology.** Let $\mathcal{G} = (\Gamma, \ \alpha, \ \theta)$ be a GKM graph. First we put the set of generators of $\mathfrak{t}^*_{\mathbb{Z}}$ by $\{\alpha_1, \ \cdots, \ \alpha_n\}$. Then we can consider the equivariant cohomology of a point as follows:

$$H^*_T(pt) = H^*(BT) \simeq \mathbb{Z}[\alpha_1, \ \cdots, \ \alpha_n].$$

Definition 4.6 (graph equivariant cohomology)**.** We define the ring $H^*_T(\mathcal{G})$ of $\mathcal{G} = (\Gamma, \ \alpha, \ \theta)$ as follows:

$$H^*_T(\mathcal{G}) = \{f : \mathcal{V}^\Gamma \to H^*_{T^n}(pt) \mid f(p) - f(q) \equiv 0 \pmod{\alpha(pq)}\},$$

where $pq \in E^\Gamma$ is an edge and $\alpha(pq) \in \mathfrak{t}^*_{\mathbb{Z}} \simeq H^2(BT)$. We call $H^*_T(\mathcal{G})$ a *graph equivariant cohomology.* We also call the relation $f(p) - f(q) \equiv 0 \pmod{\alpha(pq)}$ a *congruence relation* of f.

Example 4.7. Let $\mathcal{G}$ be the GKM graph in Example 4.5. Let us look at the element in $H^*_T(\mathcal{G})$. For the GKM graph in Figure 6, the following map $f : \mathcal{V}^\Gamma \to \mathbb{Z}[\alpha, \beta]$ is one of the elements in $H_T^*(\mathcal{G})$:

$$f(p) = \alpha\beta;$$
$$f(q) = \alpha\beta^2 + \alpha;$$
$$f(r) = \alpha^2\beta + \beta,$$

because f satisfies the congruence relation.

5. Relations with toric geometry

A *quasitoric manifold* is the equivariantly formal torus manifold, i.e., $H^{odd}(M) = 0$ such that its orbit space is combinatorially same as a simple polytope P. Because a quasitoric manifold is an equivariantly formal torus manifold, a quasitoric manifold is a GKM manifold by Example 3.9. Hence, we can define the GKM graph $\mathcal{G}$ of the quasitoric manifolds. In this section, we quickly review the generators and their relations of its graph equivariant cohomology $H_T^*(\mathcal{G})$.

By the definition of a quasitoric manifold, its 1-skelton corresponds to the 1-skelton of its orbit polytope, i.e., for the orbit projection $p : M \to P$, the inverse of 1-skelton $p^{-1}\Gamma(P)$ is M_1 (where $\Gamma(P)$ is the 1-skelton of P). We can easily see that $\Gamma(P)$ is an n-valent graph if $\dim M = 2n$. Due to [MP06], we see that the generator of $H_T(\mathcal{G})$ is the set of all *facets* of $\mathcal{G}$, i.e., the set of all $(n-1)$-valent GKM subgraphs and their relation is the ideal generated by $\prod_{H\subset\Gamma} \tau_H$ such that $\cap_{H\subset\Gamma} H = \emptyset$, where $\tau_H \in H_T^*(\mathcal{G})$ is the element which corresponds to the facet H.

Example 5.1. Recall Example 4.5. This is defined by the quasitoric manifolds $\mathbb{C}P(2)$. We look at its generators and multiplications in $H_T^*(\mathcal{G})$.

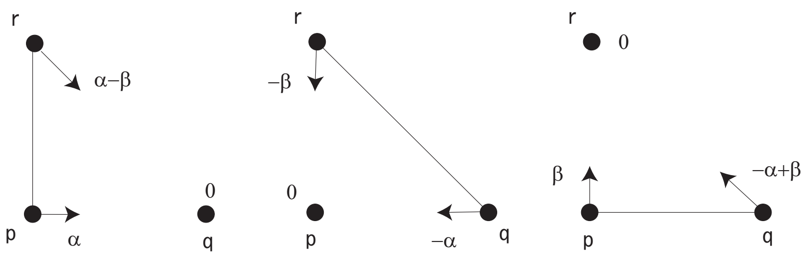

Figure 7. The generators τ_{pr}, τ_{qr}, τ_{pq} (from left) of the GKM graph in Example 4.5.

In Figure 7, we show the generators in the GKM graph in Example 4.5. There are just three facets: pr, qr, pq, i.e., 1-valent GKM subgraphs in $\mathcal{G}$ of Example 4.5. The corresponding elements of these facets in $H_T^*(\mathcal{G})$ is the elements shown in Figure 7, i.e.,

$$\tau_{pr}(p) = \alpha, \quad \tau_{pr}(q) = 0, \quad \tau_{pr}(r) = \alpha - \beta;$$
$$\tau_{qr}(p) = 0, \quad \tau_{qr}(q) = -\alpha, \quad \tau_{qr}(r) = -\beta;$$
$$\tau_{pq}(p) = \beta, \quad \tau_{pq}(q) = -\alpha + \beta, \quad \tau_{pq}(r) = 0,$$

and their relations are

$$\begin{array}{lll} \tau_{pr}\tau_{qr}(p) = 0, & \tau_{pr}\tau_{qr}(q) = 0, & \tau_{pr}\tau_{qr}(r) = -\beta(\alpha - \beta); \\ \tau_{pr}\tau_{pq}(p) = \alpha\beta, & \tau_{pr}\tau_{pq}(q) = 0, & \tau_{pr}\tau_{pq}(r) = 0; \\ \tau_{qr}\tau_{pq}(p) = 0, & \tau_{qr}\tau_{pq}(q) = -\alpha(-\alpha + \beta), & \tau_{qr}\tau_{pq}(r) = 0, \end{array}$$

and

$$\tau_{pr}\tau_{qr}\tau_{pq} = 0,$$

that is, we have

$$H_T^*(\mathcal{G}) \simeq \mathbb{Z}[\tau_{pr},\ \tau_{qr},\ \tau_{pq}]/ < \tau_{pr}\tau_{qr}\tau_{pq} > \simeq H_T^*(\mathbb{C}P(2)).$$

This is the *face ring* of Δ^2.

6. Problem

Finally, we give the (principle) problem of GKM theory (also see [GZ01]).

Problem 6.1. Let $\mathcal{G}$ be a GKM graph. What is the ring structure of $H_T^*(\mathcal{G})$? That is, determine generators and their multiplicative structure.

As we see in the Section 2, GKM manifolds include equivariantly formal torus manifolds, hypertoric varieties, and Grassmanianns, e.t.c. Moreover, we have already known the independent formulations of $H_T^*(M)$ for these three classes, e.g. the *face ring* of simplicial poset for the equivariantly formal torus manifold (see [MP06] and [MMP07]), the *Stanley-Reisner ring* of unoriented matroid for hypertoric varieties (see [HP04] and [HH05]), and the *equivariant Schubert calculus* for Grassmanianns (see [KT03]). If we had the answer of Problem 6.1, then we would unite the above independent formulations!

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