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A SURVEY ON HAMILTONIAN CIRCLE ACTIONS

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ABSTRACT. In this survey article we briefly review known results about Hamiltonian circle actions on symplectic manifolds.

In this article every manifold is assumed to be compact and connected unless otherwise stated. Let (M^{2n}, ω) be a symplectic manifold. Consider the S^1 action on M. We assume that the action is effective, that is, each element of S^1 gives a different diffeomorphism. The action ϕ_t is called *symplectic* if $\phi_t^*\omega = \omega$. This is equivalent to that $i_X\omega := \omega(X, \cdot)$ is closed since ω is closed. Here X is the vector field generated by the circle action. If $i_X\omega$ is exact, the action is called *Hamiltonian*. Therefore, the Hamiltonian action is symplectic and we want to know when the converse is true.

There are some conditions which guarantee that the symplectic circle action becomes Hamiltonian. The trivial case is given by the condition that the first Betti number is zero. Before considering non-trivial cases we state the necessary condition that the action is Hamiltonian.

When the circle action is Hamiltonian, there is a function H satisfying the equation $i_X\omega = dH$. It is called a *moment map* of the action. Since the symplectic form is nondegenerate, the critical point set is exactly the set of points fixed by the action. The moment map has an extremum, so the fixed point set is not empty. Now we can think about the converse.

Question 1. *Consider the symplectic S^1 action on the symplectic manifold. If the fixed point set is not empty, does this imply that the action is Hamiltonian?*

This is not true in general. However, in some special cases the condition that the fixed point set is non-empty is sufficient that the action is Hamiltonian. In 1959, Frankel [2] proved that it is true for Kähler case. Later Ono [5] generalized this result by replacing the Kähler condition by the condition that $\omega^{n-1} : H^1(M) \to$

$H^{2n-1}(M)$ is an isomorphism. In Kähler case, this condition is guaranteed by the hard Lefschetz theorem. The idea of the proof is as follows; if the action has a fixed point, any orbit of the action is contractible. By showing that $[i_X\omega^n]$ is the dual of the class represented by the circle orbit, he concluded $[i_X\omega]$ is zero.

In 1988, McDuff [3] proved that the answer is also yes for 4 dimensional case. For non-Hamiltonian case we can still define, rationalizing the symplectic form if necessary, the circle valued moment map, call a *generalized moment map*. She analyzed the change of the Euler number as one passes the critical level. The Euler number changes strictly monotonically, so the image of the generalized moment map cannot be the whole circle. She also proved that this is not the case for higher dimensions. She constructed a 6 dimensional symplectic manifold with non-Hamiltonian symplectic circle action with non-empty fixed point set. In her example, the fixed sets are tori. But if we impose further condition that the fixed point set is isolated, there are no known examples.

The action is called *semi-free* if the action is free on the complement of the fixed point set. Tolman and Weitsman [6] proved that when the circle action is semi-free and the fixed point set is isolated, the action is Hamiltonian if and only if the fixed set is non-empty. They used the equivariant cohomology and the localization formula to show that the generalized moment map has a local minimum. But in general case, this is still an open problem.

Question 2. *Consider the symplectic S^1 action on the symplectic manifold. Suppose that the fixed point set is non-empty and isolated. Is this action necessarily Hamiltonian?*

References

[1] M. Audin, *The topology of torus actions on symplectic manifolds.* Progress in Mathematics, 93. Birkhauser Verlag, 1991.

[2] T. Frankel, *Fixed points and torsion on Kahler manifolds.* Ann. of Math. (2) 70 1959 1–8.

[3] D. McDuff, *The moment map for circle actions on symplectic manifolds.* J. Geom. Phys. 5 (1988), no. 2, 149–160.

[4] D. McDuff and D. Salamon, *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, 1995.

[5] K. Ono, *Equivariant projective imbedding theorem for symplectic manifolds.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988), no. 2, 381–392.

[6] S. Tolman and J. Weitsman, *On semifree symplectic circle actions with isolated fixed points.* Topology 39 (2000), no. 2, 299–309.