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RICCI FLOW, RICCI SOLITON AND LIMITING SINGULARITY MODEL

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1. Introduction

In this article, we will introduce Ricci flow theory and its results. Especially, We will focus on Ricci solitons and limiting singularity model.

In 1982, Hamilton (cf. [12]) introduced the Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

to study compact three-manifolds with positive Ricci curvature. The Ricci flow, which evolves a Riemannian metric by its Ricci curvature, is a natural analogue of the heat equation for metrics. In geodesic coordinates $\{x^i\}$,

$$g_{ij}(x) = \delta_{ij} + \tfrac{1}{3}R_{ipqj}x^p x^q + O(|x|^3).$$

Hence,

$$\Delta(g_{ij}) = \Big(\sum_{k=1}^n \tfrac{\partial^2}{\partial (x^k)^2} g_{ij}\Big)(0) = \tfrac{-2}{3}R_{ij}.$$

So, the Ricci flow is[1]

$$\tfrac{\partial}{\partial t} g_{ij}(t) = 3\Delta(g_{ij}).$$

In 1995, Hamilton proved long time existence of Ricci flow solution (cf. §8 in [14]): for any smooth initial metric on a compact manifold, there exists a maximal time T on which there is a unique smooth solution to the Ricci flow for $0 \le t < T$. Either $T = \infty$ or else the curvature is unbounded as $t \to T$. This theorem is proven by Shi's the global version[2] of the derivative estimates[3]

[1]The above is described in Hamilton's paper (cf. [14])
[2]There exists a local version in [26]
[3]The similar arguments will be also found in § 5.3 Curvature blow-up at finite-time singularities of the book [29].

Theorem 1.1. ([26]) *There exist constants C_k for $k \geq 1$ such that if the curvature is bounded*

$$|\mathrm{Rm}| \leq M$$

up to time t with $0 < t \leq 1/M$, then, the covariant derivative of the curvature is bounded on $(0, 1/M]$

$$|D\mathrm{Rm}| \leq C_1 M/t^{1/2}$$

and the k^{th} covariant derivative of the curvature is bounded

$$|D^k\mathrm{Rm}| \leq C_k M/t^{k/2}.$$

Now, we will consider a general result.

2. General results

In this section, we will introduce the results about Ricci flow.

2.1. **Hamilton's results.** Hamilton introduced the normalized Ricci flow:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij} + \frac{2}{n} r g_{ij}, \tag{2.1}$$

where n=dimension of M and $r = \frac{\int_M Rdvol(g(t))}{\int_M dvol(g(t))}$.

Note that a normalized Ricci flow preserves a volume:

$$\begin{gathered}\frac{\partial}{\partial t}\int_M dvol(g(t)) = \int_M \frac{\partial}{\partial t}\sqrt{\det g(t)}dX \\ = \int_M \tfrac{1}{2}(g_t)_{ij}g^{ij}\sqrt{\det g(t)}dX = \int_M \tfrac{1}{2}(-2R_{ij} + \tfrac{2}{n}g_{ij})g^{ij}dvol(g(t)) \\ = -\int_M Rdvol(g(t)) + r\int_M dvol(g(t)) = 0.\end{gathered}$$

Thus

$$\int_M dvol(g(t)) \equiv \text{const.}$$

In 1988, Hamilton (cf. [13]) showed that if M is not diffeomorphic to the 2-sphere S^2, then any metric g converges to a constant curvature metric under (2.1). Furthermore, if M is diffeomorphic to the 2-sphere S^2, then any metric g with positive sectional curvature converges to a constant curvature metric under (2.1). Chow (cf. [4]) refined this result: if g is any metric on S^2, then under Hamilton's Ricci flow (2.1), the sectional curvature becomes positive in a finite time. Note that in dimension 3, Hamilton (cf. [12]) proved the following: let M be a compact Riemannian manifold with strictly positive Ricci curvature. then, M also admits a metric of constant positive sectional curvature under (2.1).

2.2. **2-positive curvature operator.** Differently to the preceeding subsection, we will comment about general dimensions.

Definition 2.1. A manifold has 2-positive curvature operator if the sum of the lowest two eigenvalue of a curvature operator is positive.

Böhm and Wilking proved the following by constructing a pinching family. This technique inspires Theorem 2.7.

Theorem 2.2. *(cf. [3]) On a compact manifold, the normalized Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.*

2.3. **Sphere theorem.** In this subsection, we will introduce results about sphere theorem.

Let M is a complete, simply-connected, n-dimensional Riemannian manifold with sectional curvature K_M. The sphere theorem was conjectured by Rauch and proven by Berger and Klingenberg around 1960 by comparison techniques: if $K_M \in (1, 4]$, then M is homeomorphic to the n-sphere. If we relax the assumption on the strict lower bound in the above, then we have the following rigidity result.

Theorem 2.3. *(Berger's Rigidity theorem) Let M is a complete, simply-connected, n-dimensional Riemannian manifold with $\frac{1}{4} \leq K_M \leq 1$. Then either M is homeomorphic to S^n or M is isometric to a symmetric space.*

Definition 2.4. We say that the sectional curvatures of M is strict pointwise δ-pinched $(0 < \delta \leq 1)$ if for any $p \in M$,

$$0 < K_1 < \tfrac{1}{\delta} K_2),$$

where K_i is a sectional curvature for the plane $P_i \subseteq T_pM$.

Let $\underline{K}(x) = \inf_{\sigma \subseteq T_xM} K(\sigma)$ and $\overline{K}(x) = \sup_{\sigma \subseteq T_xM} K(\sigma)$, where σ is a any two plane $\subseteq T_xM$.

Under the condition $\underline{K} > 0$ and M is compact, note that M is strict pointwise δ-pinched if and only if $\delta\overline{K} < K \leq \overline{K}$ (or $\underline{K} \leq K < \frac{1}{\delta}\underline{K}$)

Definition 2.5. A manifold has a positive isotropic curvature (=PIC) (resp. non-negative isotropic curvature or weakly PIC) if

$K_{13} + K_{14} + K_{23} + K_{24} - 2R_{1234} > 0$ (resp. $K_{13} + K_{14} + K_{23} + K_{24} - 2R_{1234} \geq 0$),

where $K_{ij} = R_{ijji}$ is a sectional curvature for a plane generated by $\{\overrightarrow{e}_i, \overrightarrow{e}_j\}$ and $\{\overrightarrow{e}_i\}$ is a frame.

Theorem 2.6. *(Micallef and Moore, 1988* [22]*) A compact, simply connected manifold with positive isotropic curvature is homeomorphic to S^n. Moreover, they observed that a strict pointwise 1/4-pinched condition implies a positive isotropic curvature condition.*

By showing that a positive isotropic curvature condition is preserved by Ricci flow, Brendle and Schoen proved the following.

Theorem 2.7. *(Differential Sphere Theorem-Brendle and Schoen, 2007* [2]*)*

Let M be a compact Riemannian manifold of dimension $n \geq 4$ with strictly pointwise 1/4-pinched sectional curvatures. Then M admits a metric of constant curvature and therefore is diffeomorphic to a spherical space form.

2.4. **Perelman's work.** Recently, Perelman solved Poincaré conjecture (cf. Section 3.2 in [19], [25]). Note that one of his most important results is the following (cf. [24], [20]).

Theorem 2.8. *(No local collapsing theorem, 2002* [24]*) Let M be a compact n-dimensional manifold. If g(t) is a Ricci flow solution on M that exists for $t \in [0, T)$, with $T < \infty$, then every $\rho > 0$ there is a $\kappa > 0$ with the following property: suppose that $r \in (0, \rho)$ and let $B_t(x, r)$ be a metric r-ball in a time-t slice. if the sectional curvatures on $B_t(x, r)$ are bounded in absolute value by $\frac{1}{r^2}$, then, the volume of $B_t(x, r)$ is bounded below by κr^n.*

This theorem can be proven by using the Perelman's reduced volume function (cf. [24]):

$$\widetilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} \exp\Big(- \frac{1}{2\sqrt{\tau}} L(x, \tau)\Big) dV_\tau, \tag{2.2}$$

where dV_τ denotes the volume element with respect to the metric $g_{ij}(\tau)$. The detailed proof is given by Cao and Zhu (cf. [10]). Now, we will characterize Perelman's reduced volume. Consider the manifold $\widetilde{M} = M \times \mathbb{S}^N \times \mathbb{R}^+$ with the following metric:

$$\begin{aligned} \widetilde{g}_{ij} &= g_{ij}, \\ \widetilde{g}_{\alpha\beta} &= \tau g_{\alpha\beta}, \\ \widetilde{g}_{oo} &= \tfrac{N}{2\tau} + R, \\ \widetilde{g}_{i\alpha} = \widetilde{g}_{io} &= \widetilde{g}_{o\alpha} = 0, \end{aligned}$$

where i, j are coordinate indices on M; α, β are coordinate indices on $\mathbb{S}^N$; and the coordinate τ on $\mathbb{R}^+$ has index o. Let $\tau = T - t$ for some fixed constant T. Then, g_{ij} will evolve with τ by the backward Ricci flow $\frac{\partial}{\partial\tau} g_{ij} = 2R_{ij}$. The metric $g_{\alpha\beta}$ on $\mathbb{S}^N$ is a metric with constant sectional curvature $\frac{1}{2N}$. $S_{\widetilde{M}}(r)$=a metric sphere in $\widetilde{M}$

of radius r. Then,

$$\frac{\mathrm{Vol}(S_{\widetilde{M}}(\sqrt{2N\tau}))}{\mathrm{Vol}(S_{\mathbf{R}^{n+N+1}}(\sqrt{2N\tau}))} \approx \mathrm{const}\cdot N^{-\frac{n}{2}}\int_M (4\pi\tau)^{-\frac{n}{2}}\exp\Big(-\frac{1}{2\sqrt{\tau}}L(x,\tau)\Big)dV_\tau$$
$$= \mathrm{const}\cdot N^{-\frac{n}{2}}\widetilde{V}(\tau).$$

Since the Ricci curvature of $\widetilde{M}$ is zero (modulo N^{-1}), the Bishop volume comparison theorem (cf. p. 68–69 in [9]) then suggests that the integral $\widetilde{V}(\tau)$ should be nonincreasing[4] in τ.

Now, we will consider how no local collapsing theorem play a role in Ricci flow theory. Recall that for a Ricci flow $g(t)$ on a closed manifold M, over a maximal time interval $[0,T)$ with $T<\infty$, we found appropriate sequences $\{p_i\}\subset M$ and $t_i \uparrow T$, with

$$|\mathrm{Rm}|(p_i,t_i) = \sup_{x\in M,\ t_2\in[0,t_i]}|\mathrm{Rm}|(x,t)\to\infty,$$

and defined blown-up Ricci flows

$$g_i(t) := |\mathrm{Rm}|(p_i,t_i)g(t_i+\tfrac{t}{|\mathrm{Rm}|(p_i,t_i)}).$$

Note that by Theorem 2.8 and Chapter 8 in [29], $\inf_i \mathrm{inj}(M_i,g_i(0),p_i)$ has a uniform lower bound. So, by applying the following theorem, we will obtain the limit of $\{g_i(t)\}$.

Theorem 2.9. *(Compactness theorem of Ricci flows, Hamilton, 1995* [15]*) Let M_i be a sequence of manifolds of dimension n, and let $p_i\in M_i$ for each i. Suppose that $g_i(t)$ is a sequence of complete Ricci flows on M_i for $t\in(a,b)$, where $-\infty\le a<0<b\le\infty$. Suppose that*

(1) $\sup_i \sup_{x\in M_i, t\in(a,b)}|\mathrm{Rm}(g_i(t))|(x)<\infty$
(2) $\inf_i \mathrm{inj}(M_i,g_i(0),p_i)>0$.

Then there exist a manifold M of dimension n, a complete Ricci flow $g(t)$ on M for $t\in(a,b)$, and a point $p\in M$ such that, after passing to a subsequence in i,

$$(M_i,g_i(t),p_i)\to(M,g(t),p),$$

as $i\to\infty$.

Here, $(M_i,g_i(t),p_i)$ converges to $(M,g(t),p)$ in the following sense.

Definition 2.10. (Geometrical limit [30]) A sequence (M_n,g_n,p_n) of pointed d-dimensional connected complete Riemannian manifolds is said to converge geometrically to another pointed d-dimensional connected complete Riemannian manifold $(M_\infty,g_\infty,p_\infty)$ if there exists a sequence $V_1\subset V_2\subset\cdots$ of connected neighborhoods

[4]Note that this is not a rigorous proof.

of p_∞ increasing to M_∞ (i.e. $\bigcup_n V_n = M_\infty$) and a sequence of smooth embeddings $\phi_n : V_n \to M_n$ mapping p_∞ to p_n such that

(1) The closure of each V_n is compact and contained in V_{n+1} (note that this implies that every compact subset of M_∞ will be contained in V_n for sufficiently large n);
(2) The pullback metric $\phi_n^* g_n$ converges in the $C^\infty_{loc}(M_\infty)$ topology to g_∞ (i.e. all derivatives of the metric converge uniformly on compact sets).

3. Open problems

In this section, we will comments some problems related with Ricci flow theory.

3.1. **Eigenvalue.**

It is interesting to see the change of eigenvalue of some operator along Ricci flow.

Theorem 3.1. *(cf. [21]) Let $g(t)$, $t \in [0,T)$, be a solution to the Ricci flow on a closed Riemannian manifold M^n. Assume that there is a C^1-family of smooth functions $f(t) > 0$, which satisfy*

$$\lambda(t)f(t) = -\Delta_{g(t)} f(t) + 1/2 R_{g(t)} f(t)$$
$$\int_M f^2(t) d\mu_{g(t)} = 1$$

where $\lambda(t)$ is a function of t only. Then

$$\begin{aligned} 2\frac{d}{dt}\lambda(t) &= 4\int R_{ij}\nabla^i f \nabla^j f d\mu + 2\int |Rc|^2 f^2 d\mu \\ &= \int |R_{ij} + \nabla_i\nabla_j\varphi|^2 e^{-\varphi} d\mu + \int |Rc|^2 e^{-\varphi} d\mu \\ &\geq 0, \end{aligned}$$

where φ satisfies $e^{-\varphi} = f^2$.

Let λ_1 be the smallest positive eigenvalue of the Laplacian.

Conjecture 3.2. *(cf. [6]) Let (S^2, g_0) be a topological sphere endowed with a smooth metric normalized to volume one and let $g(t)$ be the unique solution of the normalized Ricci flow*

$$\frac{\partial}{\partial t} g = (r - R)g$$
$$g(0) = g_0.$$

Then $\lambda_1(g(t))$ is increasing for all $t \in [0,\infty)$ and converges to $\lambda_1(g_S) = 8\pi$.

3.2. **Type I singularities on 3-manifolds.** Consider a solution $g_{ij}(x,t)$ of the Ricci flow on $M \times [0,T)$, $T \leq \infty$, where either M is compact or at each time t the metric $g_{ij}(\cdot,t)$ is complete and has bounded curvature. We say that $g_{ij}(x,t)$ is a maximal solution if either $T = +\infty$ or $T < +\infty$ and $|\mathrm{Rm}|$ is unbounded as $t \to T$. Denote by

$$K_{\max}(t) = \sup_{x\in M}|\mathrm{Rm}(x,t)|_{g_{ij}(t)}.$$

We say that $\{(x_k,t_k) \in M \times [0,T)\}$, $k = 1,\ 2,\ \cdots$, is a sequence of (almost) maximum points if there exist positive constants c_1 and $\alpha \in (0,1]$ such that $|\mathrm{Rm}(x_k,t_k)| \geq c_1 K_{\max}(t)$, $t \in [t_k - \frac{\alpha}{K_{\max}(t_k)}, t_k]$ for all k. We say that the solution satisfies injectivity radius condition if for any sequence of (almost) maximum points $\{(x_k,t_k)\}$, there exists a constant $c_2 > 0$ independent of k such that

$$\mathrm{inj}(M, x_k, g_{ij}(t_k)) \geq \frac{c_2}{\sqrt{K_{\max}(t_k)}}$$

for all k.

According to Hamilton (cf. Theorem in [16]), we classify maximal solutions into three types; every maximal solution is clearly of one and only one of the following three types:
Type I: $T < +\infty$ and $\sup_{t\in[0,T)}(T-t)K_{\max}(t) < +\infty$
Type II:
(a) $T < +\infty$ but $\sup_{t\in[0,T)}(T-t)K_{\max}(t) = +\infty$
(b) $T = +\infty$ but $\sup_{t\in[0,T)} tK_{\max}(t) = +\infty$
Type III:
(a) $T = +\infty$, $\sup_{t\in[0,T)} tK_{\max}(t) < +\infty$, and $\mathrm{limsup}_{t\to+\infty} tK_{\max}(t) > 0$
(b) $T = +\infty$, $\sup_{t\in[0,T)} tK_{\max}(t) < +\infty$, and $\mathrm{limsup}_{t\to+\infty} tK_{\max}(t) = 0$

It seems that Type III (b) is not compatible with the injectivity radius condition unless it is a trivial flat solution.

For each type of solution, we define a corresponding type of limiting singularity model. A solution $g_{ij}(x,t)$ to the Ricci flow on the manifold M, where either M is compact or at each time t the metric $g_{ij}(\cdot,t)$ is complete and has bounded curvature, is called a singularity model if it is not flat and of one of the following three types:
Type I: The solution exists for $t \in (-\infty, \Omega)$ for some constant Ω with $0 < \Omega < +\infty$ and

$$|\mathrm{Rm}| \leq \Omega/(\Omega - t)$$

everywhere with equality somewhere at $t = 0$
Type II: The solution exists for $t \in (-\infty, +\infty)$ and $|\mathrm{Rm}| \leq 1$ everywhere with equality somewhere at $t = 0$

Type III: The solution exists for $t \in (-A, +\infty)$ for some constant A with $0 < A < +\infty$ and

$$|\mathrm{Rm}| \leq A/(A+t)$$

everywhere with equality somewhere at $t = 0$.

We expect that each singularity converges to a corresponding singularity model. Note that there exists a following result.

Theorem 3.3. *For any maximal solution to the Ricci flow which satisfies the injectivity radius condition and is of Type I, II(a), (b), or III(a), there exists a sequence of dilations of the solution along (almost) maximum points which converges in the C^∞_{loc} topology to a singularity model of the corresponding type.*

We will sketch the proof in Type I-case (See for the detailed proof p. 293-294 in [10]): we consider a maximal solution $g_{ij}(x,t)$ on $M \times [0,T)$ with $T < +\infty$ and

$$\Omega = \ \mathrm{limsup}_{t \to T}(T-t)K_{\max}(t) < +\infty.$$

Choose a sequence of points x_k and times t_k such that $t_k \to T$ and

$$\lim_{k \to \infty}(T-t_k)|\mathrm{Rm}(x_k,t_k)| = \Omega.$$

Denote by

$$\epsilon_k = \frac{1}{\sqrt{|\mathrm{Rm}(x_k,t_k)|}}.$$

We translate in time so that t_k becomes 0, dilate in space by the factor ϵ_k^{-2} and dilate in time by ϵ_k^2 to get

$$\widetilde{g}_{ij}^{(k)}(\cdot,\widetilde{t}) = \epsilon_k^{-2} g_{ij}(\cdot, t_k + \epsilon_k^2 \widetilde{t}),\ \widetilde{t} \in [-t_k/\epsilon_k^2, (T-t_k)/\epsilon_k^2).$$

Then, $\widetilde{g}_{ij}^{(k)}(\cdot,\widetilde{t})$ is still a solution to the Ricci flow which exists on the time interval $[-t_k/\epsilon_k^2, (T-t_k)/\epsilon_k^2)$, where

$$t_k/\epsilon_k^2 = t_k|\mathrm{Rm}(x_k,t_k)| \to +\infty$$

and

$$(T-t_k)/\epsilon_k^2 = (T-t_k)|\mathrm{Rm}(x_k,t_k)| \to \Omega.$$

For any $\epsilon > 0$ we can find a time $\tau < T$ such that for $t \in [\tau, T)$,

$$|\mathrm{Rm}| \leq (\Omega + \epsilon)/(T-t).$$

Then for $\widetilde{t} \in [(\tau - t_k)/\epsilon_k^2, (T-t_k)/\epsilon_k^2)$,

$$|\widetilde{\mathrm{Rm}}^{(k)}| = \epsilon_k^2|\mathrm{Rm}| \leq (\Omega+\epsilon)/(T-t)|\mathrm{Rm}(x_k,t_k)|) \to (\Omega+\epsilon)/(\Omega - \widetilde{t}),\ \text{as} k \to \infty.$$

By an argument in Subsection 2.4, a curvature of $\widetilde{g}_{ij}^{(\infty)}$ satisfies the bound

$$|\widetilde{\mathrm{Rm}}^{(\infty)}| \leq \Omega/(\Omega - \widetilde{t})$$

everywhere on $\widetilde{M} \times (-\infty, \Omega)$ with the equality somewhere at $\widetilde{t} = 0$.

Hence, by Theorem 3.3 the classification of manifolds is reduced to the classification of singularity models in some sense. Now, since the classification of manifolds in Riemannian geometry is very important, we will state several problems and facts related to singularity models. Notice that there exists a lot of problems about classification of solutions on 2-dimensional manifolds.

(Problem 1, p. 337 in [9]) Suppose that $(M^2, g(t))$ is a complete ancient[5] solution with bounded curvature (the bound may depend on time). If $R[g(t)] > 0$ everywhere, then $(M^2, g(t))$ is homothetic to one of the following solutions: the Cigar soliton, the constant curvature shrinking S^2 or $\mathbf{RP}^2$ and the Rosenau solution or its $\mathbf{Z}_2$ quotient.

The followings are open problems. Are the following true or false ? (cf. p. 380-381 in [9])

(1) Type I ancient solution with $n \geq 3$ and bounded $\mathrm{Rm}(g(t)) > 0$ is compact.

(2) Noncompact Type II ancient solution with $n \geq 3$ and $\mathrm{Rm}(g(t)) > 0$ is a steady gradient Ricci soliton.

Hamilton [5] has shown that for any singularity there exists a sequence of points and times and a corresponding limit solution which is either a shrinking spherical space form $(S^3/\Gamma, g(t))$ or a quotient of a shrinking cylinder $((S^2 \times \mathbf{R})/\Gamma, (g(t) \times h(0))/\Gamma)$, which are the two model Type I singularities. The following question appears to be still open.

(Problem 2) Can one classify all Type I limits on 3-manifolds ?

Perhaps we may conjecture that the above limits are the only possibilities. Chow gives a partial result on the above problem (See [8]). But we will introduce another approach. To solve the problem, we suffice to show the uniqueness of its corresponding limit solution $g(t)$ for any sequence. We[6] expect that under some assumption on $g(t)$ the uniqueness is followed. Note that the integrability condition is used to solve some problems (cf. Chapter 12 in [1], [27], and [28])

Definition 3.4. We will say that $g(0)$ is integrable if for any solution $\mathbf{a}$ of a linearized deformation equation

$$\mathcal{D}_{g(0)}F(\mathbf{a}) = 0,$$

there exists a path $F(g(t)) \equiv 0$ for $t \in (-\epsilon, \epsilon)$ such that

$$\tfrac{d}{dt}|_{t=0}\, g(t) = \mathbf{a}.$$

Sesum considered in her paper the case, F=Rc (cf. [27]).

[5] An ancient solution is a solution defined on an interval of the form $(-\infty, \omega)$.

[6] This is an author's private thinking

3.3. **gap theorem.** The following theorem is interesting.

Theorem 3.5. *(Gap theorem for the Ricci flow, 2008* [31]*)*

There exists $\varepsilon_n > 0$ which satisfies the following: let $(M^n, g(\tau))$, $\tau \in [0, \infty)$ be a complete ancient solution to the backward Ricci flow on an n-manifold M with Ricci curvature bounded below. Suppose that the asymtotic limit of the reduced volume $\lim_{\tau\to\infty} \widetilde{V}_{(p,0)}(\tau) \geq 1 - \varepsilon_n$ for some $p \in M$. Then, $(M^n, g(\tau))$ is the Gaussian soliton, i.e., isometric to the Euclidean space $(\mathbf{R}^n, g_E)$ for all $\tau \in [0, \infty)$.

In fact, by regarding Ricci flat manifold as an ancient solution to the backward Ricci flow, the above theorem recover the following result, which is the motivation.

Theorem 3.6. *(Anderson* [31]*) There exists $\varepsilon_n > 0$ which satisfies the following: let (Mn, g) be an n-dimensional complete Ricci flat Riemannian manifold. Suppose that the asymptotic volume ratio $\nu(g) := \lim_{r\to\infty} VolB(p, r)/\omega_n r^n$ of g is greater than $1 - \varepsilon_n$. Here ω_n stands for the volume of the unit ball in the Euclidean space $(\mathbf{R^n}, g_E)$. Then (M^n, g) is isometric to $(\mathbf{R^n}, g_E)$.*

We will close this subsection by introducing the Gaussian soliton. The Gaussian soliton is $(\mathbf{R}^n, g_E, f = -\frac{\varepsilon|x|^2}{4})$ (cf. p. 4-5 in [7]). Note that $R(g_E)_{ij} = 0$ and

$$\mathcal{L}_{\nabla f} g_E + \varepsilon g_E.$$

By p. 2-5 in [7], define $g(t) = \sigma(t)\varphi(t)^* g_E (= g_E)$, where $\sigma(t) = 1 + \varepsilon t$ and $\varphi(t) = \sigma^{-\frac{1}{2}} \mathrm{id}_{\mathbf{R}^n}$.

From $\frac{d}{dt}\varphi = \frac{1}{\sigma}\nabla f = \nabla f(t)$,

$$f(t) = \tfrac{1}{\sigma} f$$

Note that

$$\nabla\nabla f(t) + \frac{\varepsilon}{2\sigma} g_E = 0.$$

By Corollary 3.2.4 in [10], $f(t) = -l$ and $\tau = \frac{\sigma}{\varepsilon}$. Then

$$\widetilde{V}(\tau) = \int_{\mathbf{R}^n} (4\pi\tau)^{\frac{-n}{2}} e^{-l} dvol_\tau = \int_{\mathbf{R}^n} (4\pi\tau)^{\frac{-n}{2}} e^{-\frac{|x|^2}{4\tau}} dvol(g_E) = 1.$$

Similar arguments is in the proof of Lemma 7.25 in [23].

4. Ricci solitons

We will close this article by introducing Ricci soliton, which is one of areas in Ricci flow theory. A solution to an evolution equation which moves under a one-parameter subgroup of the symmetry group of the equation is called a soliton. Hence, a solitons are referred more commonly to as self-similar solutions in PDE sense.

Let (M^n, g_0) be closed manifold and let $(M^n, g(t))$ be a solution to the Ricci flow with initial condition $g(0) = g_0$.

Definition 4.1. The solution is called Ricci steady (resp. shrinking, expanding) breather, if for some $t > 0$ and $\alpha = 1$ (resp. $\alpha > 1$, $0 < \alpha < 1$), there exists φ such that

$$\alpha g(t) = \varphi^* g(0)$$

Definition 4.2. Ricci breathers is trivial if there exists $(\alpha(t), \varphi(t))$ such that

$$\alpha(t) g(t) = \varphi(t)^* g(0)$$

We call trivial breather a steady (resp. shrinking, expanding) Ricci soliton with respect to $\alpha(t) = 0$ (resp. $\alpha(t) > 1$, $0 < \alpha(t) < 1$).

Note that the following lemma gives another definitions of Ricci solitons.

Lemma 4.3. *The Ricci flow solution $g(t)$ is Ricci soliton if and only if the initial metric $g_0 = g(0)$ satisfies*

$$Ric(g_0) + L_X g_0 = \lambda g_0 \tag{4.1}$$

for some vector field X.

Proof. If φ_t is a one-parameter group of diffeomorphisms generated by a vector field V on M, then, the Ricci soliton is given by

$$g_{ij}(x,t) = \sigma(t) \varphi_t^* g_{ij}(x, 0),$$

where $\sigma(t) = 1 - 2\lambda t$.
In particular, the initial metric $g_{ij}(x, 0)$ satisfies the following equation:

$$Ric + L_X g = \lambda g \text{ (or } R_{ij} + g^{ij}\nabla_i X^j + g^{ij}\nabla_j X^i = \lambda g_{ij}).$$

$g(t)$ is a solution to Ricci Flow if and only if $g(0)$ satisfies (4): Let φ_t be a one-parameter group of diffeomorphisms generated by Y, where $Y = \sigma X$. Note that $Y(x, 0) = 2X(x, 0)$.

$$\begin{aligned}
&\frac{\partial}{\partial t} g(t)\\
&= -2\lambda \varphi_t^* g_{ij}(y, 0) + \sigma \frac{\partial}{\partial s}|_{s=0} \varphi_s^* \varphi_t^* g(y, 0)\\
&= -2\varphi_t^*(Ric(g(0)) + L_{X(y)} g(y, 0)) + \sigma L_{X(x)} \varphi_t^* g(y, 0)\\
&= -2Ric(g(t)) - (1-1) L_{Y(x)} \varphi_t^* g(y, 0)\\
&= -2Ric(g(t)),
\end{aligned}$$

where $y = \varphi_t(x)$. ∎

By the above lemma, (4.1) is called by the Ricci soliton equation.

Definition 4.4. A shrinking (resp. steady and expanding) Ricci soliton equation:

$$R_{ij} + g^{ij}\nabla_i X^j + g^{ij}\nabla_j X^i - \lambda g_{ij} = 0,$$

where $\lambda > 0$ (resp. $\lambda = 0$ and $\lambda < 0$).

Therefore, g_0 is a Ricci soliton if it satisfies a Ricci soliton equation. Especially, a Ricci soliton g_0 is a gradient Ricci soliton if X is a gradient of a function f.

To help to understand Ricci solitons, consider the case where g_0 is a gradient shrinking Ricci soliton and f is constant. Since the initial metric is Einstein with positive scalar curvature, then the metric will shrink under Ricci Flow by a time-dependent factor. So, we have

$$Ric_{ij}(x,0) = \lambda g_{ij}(x,0), \forall x \in M$$

and some $\lambda > 0$. From the definition of the Ricci tensor, one sees that

$$Ric_{ij}(x,t) = Ric_{ij}(x,0) = \lambda g_{ij}(x,0).$$

Let

$$g_{ij}(x,0) = \rho(t)^2 g_{ij}(x,0).$$

Thus, the equation (1.1) corresponds to

$$\frac{\partial(\rho(t)^2 g_{ij}(x,0))}{\partial t} = -2\lambda g_{ij}(x,0).$$

This gives the ODE

$$\frac{d\rho}{dt} = -\frac{\lambda}{\rho},$$

whose solution is given by

$$\rho(t)^2 = 1 - 2\lambda t.$$

Thus, the evolving metric g(t) shrinks homothetically to a point as $t \to T = 1/2\lambda$. Note that as $t \to T$, the scalar curvature becomes infinite like $1/(T-t)$.

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