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LARGE DEVIATION PRINCIPLE FOR UNIMODAL MAPS SATISFYING THE COLLET-ECKMANN CONDITION

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Let I be the compact interval $[0,1]$ of the real line $\boldsymbol{R}$. We denote by $\mathscr{M}$ the space of the Borel probability measures on I equipped with the weak topology and by m Lebesgue measure. For a map $f : I \to I$ non-singular with respect to m we say that *the large deviation principle* holds if there is an upper semicontinuous function $q : \mathscr{M} \to [-\infty, 0]$, called *the rate function,* satisfying

$$\liminf_{n\to\infty} \frac{1}{n} \log m\{x \in I : \delta_x^n \in \mathscr{G}\} \geq \sup_{\mu\in\mathscr{G}} q(\mu) \quad \text{for each open set} \quad \mathscr{G} \subset \mathscr{M},$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log m\{x \in I : \delta_x^n \in \mathscr{C}\} \leq \max_{\mu\in\mathscr{C}} q(\mu) \quad \text{for each closed set} \quad \mathscr{C} \subset \mathscr{M},$$

respectively, where $\delta_x^n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \in \mathscr{M}$ denotes the empirical distribution along the orbit of x. For a piecewise expanding map having some mixing property it is known by Takahashi [8, 9] that the large deviation principle holds, and the rate function coincides with the free energy mentioned later.

We consider large deviations for smooth unimodal maps. An *unimodal map* is a C^1 map $f : I \to I$ such that $f(0) = f(1) = 0$, and the derivative f' is positive on the interval $[0, c)$ and negative on the interval $(c, 1]$ for some $c \in (0,1)$. The point c is called *the critical point* of f. The critical point c is *non-flat* if there exists an integer $l > 1$ and a C^1 function $M : [0,1] \to (0,\infty)$ such that $|f'(x)| = M(x)|x - c|^{l-1}$ holds for all $x \in [0,1]$. An *S-unimodal map* is a C^2 unimodal map f satisfying the following conditions:

(1) The critical point c is non-flat;
(2) $|f'|^{-1/2}$ is convex on the intervals $[0, c)$ and $(c, 1]$;
(3) $|f'(0)| > 1$.

We say that an S-unimodal map f satisfies *the Collet-Eckmann condition* if

$$\liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(f(c))| > 0.$$

For some fundamental properties of S-unimodal maps we refer the reader to [5] and [6].

Let f be an S-unimodal map satisfying the Collet-Eckmann condition. Considering the renormalization we assume that f satisfies the following topological

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mixing condition: for any non-trivial interval $K \subset I$ there is an integer $k \geq 0$ such that $f^k(K) \supset [f^2(c), f(c)]$, where c is the critical point of f. Then there is a unique f-invariant $\nu_0 \in \mathscr{M}$ which is absolutely continuous with respect to Lebesgue measure, and (f, ν_0) has exponential decay of correlations. And then the central limit theorem holds [3, 10]. It is also known by Keller and Nowicki [3] the large deviation theorem below. Let assume that a continuous function $\varphi : I \to \boldsymbol{R}$ of bounded variation satisfies $\sigma_\varphi^2 := \int {\varphi_0}^2 \, d\nu_0 + 2\sum_{n=1}^{\infty} \int \varphi_0 \cdot (\varphi_0 \circ f^n) \, d\nu_0 > 0$ where $\varphi_0 := \varphi - \int \varphi \, d\nu_0$. Then

$$\alpha(\varepsilon) := \lim_{n\to\infty} \frac{1}{n} \log m \left\{ x \in I : |\frac{1}{n} S_n\varphi(x) - \int \varphi \, d\nu_0| > \varepsilon \right\} < 0$$

holds for small $\varepsilon > 0$, where $S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(f^i(x))$. The author [2] established the large deviation principle, and determined the rate functions for this class of unimodal maps. To state this result precisely, we indicate by $\mathscr{M}_f$ the set of the f-invariant Borel probability measures on I, and define *the free energy* $F : \mathscr{M} \to \boldsymbol{R} \cup \{-\infty\}$ by

$$F(\mu) := \begin{cases} h_\mu(f) - \int \log |f'| d\mu & \text{if } \mu \in \mathscr{M}_f, \\ -\infty & \text{otherwise,} \end{cases}$$

where $h_\mu(f)$ denotes the metric entropy of μ for f. Then F is non-positive by the Ruelle inequality [7], and $F(\mu) = 0$ iff $\mu = \nu_0$ [4]. Moreover, the stability of the free energy [1] asserts that $F(\mu_n) \to 0$ $(n \to \infty)$ implies $\mu_n \to \nu_0$ $(n \to \infty)$ on $\mathscr{M}$. The main result of [2] is the following:

Theorem . *The large deviation principle holds for any S-unimodal map f satisfying the Collet-Eckmann and the topological mixing conditions mentioned above. The rate function $q : \mathscr{M} \to [-\infty, 0]$ is given by the upper regularization of the free energy, that is*

$$q(\mu) = \inf \{Q(\mathscr{G}) : \mathscr{G} \text{ is a neighborhood of } \mu \text{ in } \mathscr{M}\}$$

where

$$Q(\mathscr{G}) = \sup_{\nu\in\mathscr{G}} F(\nu).$$

Notice that the upper semi-continuity of the free energy is not guaranteed to an S-unimodal map satisfying the Collet-Eckmann condition [1]. Thus the free energy itself is not the rate function in our theorem. This is a difference of large deviations between piecewise expanding maps and smooth unimodal maps.

Let $f : I \to I$ be as in the theorem above and $\varphi : I \to \boldsymbol{R}$ a continuous function. Here φ is not necessary to be of bounded variation. Put

$$v_* := \inf_{x\in I} \liminf_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \min_{\mu\in\mathscr{M}_f} \int \varphi \, d\mu$$

and

$$v^* := \sup_{x\in I} \limsup_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \max_{\mu\in\mathscr{M}_f} \int \varphi \, d\mu,$$

respectively. Then it follows immediately from the theorem that:

Corollary 1 (The contraction principle).

$$\begin{aligned}\sup\left\{F(\mu): a<\int\varphi\,d\mu<b\right\} &\le \liminf_{n\to\infty}\frac{1}{n}\log m\left\{x\in I: a<\frac{1}{n}S_n\varphi(x)<b\right\}\\ &\le \limsup_{n\to\infty}\frac{1}{n}\log m\left\{x\in I: a\le\frac{1}{n}S_n\varphi(x)\le b\right\}\\ &\le \lim_{\varepsilon\to 0}\sup\left\{F(\mu): a-\varepsilon<\int\varphi\,d\mu<b+\varepsilon\right\}\end{aligned}$$

holds for any $a, b \in \boldsymbol{R}$. In particular, if $a \neq v^*$ and $b \neq v_*$ then

$$\lim_{n\to\infty}\frac{1}{n}\log m\left\{x\in I: a<\frac{1}{n}S_n\varphi(x)<b\right\}=\sup\left\{F(\mu): a<\int\varphi\,d\mu<b\right\}.$$

As a consequence we get a formula that:

$$\alpha(\varepsilon)=\sup\left\{F(\mu): \left|\int\varphi\,d\mu-\int\varphi\,d\nu_0\right|>\varepsilon\right\}$$

for α in the large deviation theorem of Keller and Nowicki. Another application of the theorem is for a variational principle. *The pressure* $P(\varphi)$ of a continuous function φ for Lebesgue measure is defined by

$$P(\varphi):=\limsup_{n\to\infty}\frac{1}{n}\log\int\exp(-S_n\varphi)\,dm.$$

Then the function $P: C(I)\to\boldsymbol{R}$ is the Legendre transform of $-q$ [8], where $C(I)$ denotes the space of the continuous functions on I. Thus we obtain the following:

Corollary 2 (The variational principle of Gibbs type).

$$P(\varphi)=\sup_{\mu\in\mathcal{M}_f}\{F(\mu)-\int\varphi\,d\mu\}\quad \textit{for all}\quad \varphi\in C(I),$$

and

$$F(\mu)\le\inf_{\varphi\in C(I)}\{P(\varphi)+\int\varphi\,d\mu\}\quad\textit{for all}\quad \mu\in\mathcal{M}_f.$$

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