Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 13, Number 1, 2011, pages 1–5

NUMERICAL METHODS OF PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS FOR OPTION PRICE

YONGHOON KWON

Abstract. In this survey article, starting with the Black-Scholes equations whose solutions are the values of European options, we describe the exponential jump-diffusion model of Levy process type. This partial integro-differential model is an extension of the Black-Sholes equation combined with integral terms based on a random variable of jump size. Explicit and implicit numerical schemes of finite differences are surveyed and discussed.

1. The Black-Sholes equation

A traded *call option* (put option) is a contract which gives to its owner the right to buy (to sell) a fixed quantity of assets of a specified common stock at a fixed *exercise price* on or before a given *expiry date*. The market price of the tights to buy or sell are termed *call price* and *put price*, respectively. When the transaction involved in the option takes place the option has been *exercised*. In this article we deal with *European* and *American* call options on shares which may pay continuous dividends. The European option may be exercised only on the expiry date while the American option may be exercised on any moment before the expiry date. In any case, when the option is exercised the owner pays the exercise price and receives the underlying stock. The decision of the owner depends on the current price of the stock. Thus, if we note by C^*, S^*, and K the values on the expiry dates of the call, the share and the strike price respectively, then clearly $C^* = \max(0, S^* - E)$.

In this article we shall review several numerical methods for mathematical models whose solutions give the value of the call option in any moment prior to the expiry date. The most widely extended numerical methods for valuing derivative securities can be classified in lattice and finite difference approaches. The first ones were introduced by Cox et al. [5] and extended by Hull and Whites [7]. The finite difference approach is applied to the backward parabolic differential equation introduced by Black and Scholes [2] to model the evolution of call option price. The

2000 *Mathematics Subject Classification.* 65M06, 65M12, 91B28.
Key words and phrases. European and American options, The Black-Scholes equation, Partial integro-differential equation, Finite differences, Linear complementarity problem.

existence of a formal expression for solution of this Black-Sholes equation allows to evaluate the different numerical methods in European call options. Finite differences were suggested by Schwartz [12] for the valuation of warrants and extended by Courtadon [6] to approximate European option values. This last work is based on explicit second order schemes in time and space. Finite difference upwind scheme was proposed by C. Vazquez [14] for European and American options.

Let us consider the mathematical model proposed in Black and Scholes [4] which is widely used in real markets to obtain the current theoretical option value. The unknown of the model is the value of the call option as function of time and the market price of the underlying asset. In a wider sense, the model assumes that the present value of C of the call option depends on the current value S of the underlying asset, the strike price K, the time to expiration $T-t$ (T is the expiry date and t is the present moment), the stock volatility σ, the risk free interest rate r and the continuous dividend rate δ. The value C increases with s, $T-t$, σ, and r, and decreases with K. The dependence on dividends is more complex. Here we shall consider a continuous dividend, so it has the same effect as a negative interest rate.

let $D = (0,T) \times (0,\infty)$ be the time-stock value domain. We shall treat the following Black-Sholes equation extended with continuous dividend in [7]

$$\frac{\partial C}{\partial t}(t,s) + (r-\delta)s\frac{\partial C}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial t^2}(t,s) - rC(t,s) = 0 \quad \text{in } D, \tag{1}$$

with the pay-off function

$$C(T,s) = \max(0, s-K), \quad s \in (0,\infty), \tag{2}$$

and with the asymptotic behaviors

$$\lim_{s\to 0} C(t,s) = 0, \quad t \in [0,T] \tag{3}$$

$$\lim_{s\to\infty} C(t,s) = se^{\delta(t-T)} - Ke^{r(t-T)}, \quad t \in [0,T]. \tag{4}$$

Let $\tau = T-t$ denote the time to expiry date and $x = \log(s/s_0)$ denote the log price of the underlying asset s. Then with the domain $\Omega = [0,T] \times [-\infty,\infty]$, we can write the equations (**??**)–(**??**) as follows.

$$\tfrac{\partial C}{\partial \tau}(\tau,x) = (r-\delta)\tfrac{\partial C}{\partial x}(\tau,x) + \tfrac{1}{2}\tfrac{\partial^2 C}{\partial \tau^2}(\tau,x) - rC(\tau,x) = 0 \quad \in \Omega, \tag{5}$$

$$C(T,x) = \max(s_0 e^x - K), \quad x \in (-\infty,\infty) \tag{6}$$

$$\lim_{x\to -\infty} C(\tau,x) = 0, \quad \tau \in [0,T], \tag{7}$$

$$\lim_{x\to\infty} C(\tau,x) = s_0 e^{x-\delta\tau} - Ke^{-r\tau}, \quad \tau \in [0,T]. \tag{8}$$

2. The exponential jump-diffusion models

We assume that the log return of the stock price process s_t follows the jump-diffusion model process such that the stochastic differece equation of s_t in risk-neutral world is given by

$$\frac{ds_t}{s_{t-}} = (r-\delta-\lambda\zeta)dt + \sigma dW_t + \eta dN_t,$$

where η is a random variable of jump size from s_{t-} to $(\eta+1)s_{t-}$, and ζ is the expectation $E[\eta]$ of the random variable η, W_t is the Wiener process, and N_t is the Poisson process with density λ.

The *Merton model* has jump sizes $\ln(\eta+1)$ in the log price $\ln s_t$, associated with the density function $f(x)$ of a normal distribution given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}}. \tag{9}$$

So,

$$\zeta = e^{(\mu_j + \frac{\sigma_J^2}{2})} - 1.$$

In the *Kou model* the distribution of $\ln(\eta+1)$ is an asymmetric double exponential distribution with the density function

$$f(x) = p\lambda_+ e^{-\lambda_+ x 1_{x\geq 0} + (1-p)\lambda_-} e^{\lambda_- x} 1_{x<0}, \tag{10}$$

where $\lambda_+ > 1$, $\lambda_>0$, $0 \leq p \leq 1$, and 1_A is the indicator function of A. In this model,

$$\zeta = \frac{p\lambda_+}{\lambda_+ - 1} + \frac{(1-p)\lambda_-}{\lambda_- + 1} - 1.$$

When the undelying asset s_t follows the exponential jump-diffusion model such as the Merton and Kou models, the value of a European option u satisfies the following partial integro-differential equation (PIDE)

$$\begin{aligned} \frac{\partial u}{\partial \tau}(\tau, x) &= \frac{\sigma 62}{2} \frac{\partial^u}{\partial x^2}(\tau, x) + (r - \delta - \frac{\sigma^2}{2} - \lambda\zeta) \frac{\partial u}{\partial u} \partial x(\tau, x) - (r + \lambda(u(\tau, x) \\ & \quad + \lambda \int_{-\infty}^{\infty} u(\tau, x) f(z-x) dz, \qquad (\tau, x) \in (-\infty, \infty). \end{aligned} \tag{11}$$

For more details, see ([1],[3],[4]). The payoff function $g(\cdot)$ of u is given by

$$u(0, x) = g(x), \qquad x \in (-\infty, \infty). \tag{12}$$

In the case of European options, the payoff function of the call and put options are

$$g(x) = \max(0, s_0 e^x - K) \quad \text{and} \quad g(x) = \max(0, K - s_0 e^x), \tag{13}$$

respectively. The asymptotic behavior of the European call option is described by

$$\lim_{x\to -\infty} u(\tau, x) = 0 \quad \text{and} \quad \lim_{x\to\infty} u(\tau, x) = s_0 e^{x-\delta\tau} - Ke^{-r\tau}, \quad \tau \in [0, T]. \tag{14}$$

The value of an American option u satisfies the following linear complementarity problem (LCP)

$$\begin{cases} \frac{\partial u}{\partial \tau}(\tau, x) - \mathcal{L}u(\tau, x) \geq 0, \\ u(\tau, x) \geq g(a), \\ \left(\frac{\partial u}{\partial \tau}(\tau, x) - \mathcal{L}u(\tau, x)\right)\left(u(\tau, x) - g(x)\right) = 0, \end{cases} \tag{15}$$

for $(\tau, x) \in (0, T] \times (-\infty, \infty)$, where $\mathcal{L}u$ is the integro-differential operator of the right-hand side in (11). For more information about nemerical tretment of LCP, see Zhang [15]. The payoff function of u may be chosen as one in (13). The asymptotic behavior of the price of the American put option is

$$\lim_{x\to\infty} u(\tau, x) = K - s_0 e^x \quad \text{and} \quad \lim_{x\to\infty} u(\tau, x) = 0, \qquad \tau \in [0, T]. \tag{16}$$

3. Discretization with finite differences

In this section, we shall introduce briefly the numerical scheme proposed by Almendral and Oosterlee [1] solve the PIDE for European options. Consider a uniform mesh in space and in time, that is, let $x_i = -x^* + (i-1)h$ $(i = 1, \ldots, n)$, and $\tau_m = (m-1)k$ $(m = 1, \ldots q)$. Let $u_i^m \approx u(\tau_m, x_i)$ and $f_{ij} = f(x_j - x_i)$. The composite trapezoidal rule on $[-x^*, x*]$ gives the following approximation of the integral

$$\int_{-\infty}^{\infty} u(\tau_m, z) f(z - x_i) dz \approx \frac{h}{2}\left[u_1^m f_{i1} + u_n^m f_{in} + 2\sum_{j=2}^{n-1} u_j^m f_{ij}\right], \quad i = 1, \ldots n. \tag{17}$$

For the time variable and space variable, finite differences are used for the following approximations:

$$u_\tau(\tau_m, x_i) \approx \begin{cases} 3/2u_i^m - 2u_i^{m-1} + 1/2u_i^{m-2})/k, & \text{if } m \geq 2, \\ (u_i^m - u_i^{m-1})/k, & \text{if } m = 1, \end{cases} \tag{18}$$

$$u_{xx}(\tau_m, x_i) \approx (u_{i+1}^m - 2u_i^m + u_{i-1}^m)/h^2, \tag{19}$$

$$u_x(\tau_m, x_i) \approx (u_{i+1}^m - u_{i-1}^m)/(2h). \tag{20}$$

With the notation the vector $u^m := (u_1^m, \ldots, u_n^m)^T$ and using the finite difference schemes (17)–(20), finite difference discretization of (11)–(12) may be written in matrix for as

$$(\omega_0 I + C + D)u^m = b^m \quad \text{or} \quad \mathrm{A}u^m = b^m,$$

where the detailed elements of matrices C and D are given in Almendral and Oosterlee in [1].

Since the matrix D from the integral is dense, so is the matrix A. In order to solve for u^m the dense linear system $Au^m = b^m$, They use the regular splitting $A = Q - R$ and implement the iterative method

$$v^{l+1} = Q^{-1}Rv^l + Q^{-1}b, \quad l = 1, \ldots,, \quad v^0 = 0. \tag{21}$$

Here the regular splitting means that $Q^{-1} \geq 0$ and $R \geq 0$. The matrix Q is made by extracting the main three diagonals of A.

Since the matrix A is Toeplitz, so are Q^{-1} and R. Hence a Toeplitz matrix can be embedded into a circulant matrix C, the matrix-vector product of the iterative method can be performed by the discrete fast transform (FFT) to compute a matrix-vector product Cb where C is a circulant matrix of $n \times n$ and b is a vector. For further details on the computations of convolutions by FFT, we refer to Loan [11].

As far as the computational costs, the direct computation for Cb needs $O(n^2)$ multiplication, however FFT for convolution of a circulant matrix requires $O(n \log n)$. They performed numerical experiments for Merton's and Kou's models and reports the second-order behavior of l^∞-norms.

4. Implicit method and discussion

Due to the integro-differential operator, the PIDE is numerically challenging if it can be discetized by some fast converging implicit method. The method presented by Almendral and Oosterlee [1] is explicit and should solve linear systems of dense matrix.

For European option of PIDE, recently Kwon an Lee [9] proposed an implicit method of finite differences with three levels. The method is is an implicit method and furthermore forms linear systems with tridiagonal coefficient matrix. Hence the lienar systems can be solved directly without iterations. The second-order convergence of l^2-norm is proved. For numerical experiments for American options, Kwon an Lee [9] [10] also used the three-level implicit method with an operator splitting scheme (Ikonen and Toivanen [8]) and tested numerical computations to Merton's and Kou's models. the method also implements matrix-vector product for numerical integration by FFT. Obviously, the three-level implicit method of tridiagonal coefficient matrix performs faster and more accurate than the explicit method of dense coefficient matrix.

Finite element methods (FEM) can be used also for PIDE. Numerical performance of FEM in the sense of accuracy and computational cost for the computation of option value is generally comparable to that of finite difference methods. However, if we are really interested in computing delta-hedging $\frac{\partial u}{\partial s}$, then mixed finite element method for parabolic partial differential equation can be performed to get the same convergence order for option and delta-hedging values.

References

[1] A. Almendral, C.W. Oosterlee, Numerical vlauation of optins with jumps in the underlying, Applied Numerical Mathematics 53 (2005) 1–18.

[2] F. Black, M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy 81 (1973) 637–654.

[3] R.Cont, E.Voltchkova, A finite difference scheme for option pricing in jump diffusion and exponential Levy models, SIAM J., Numer. Anal., 43 (2005) 1596–1626.

[4] R.Cont, E.Voltchkova, Integro-differential equations for option prices in exponential Levy models, Finance. Stoch. 9 (2005) 229-325

[5] J.C. Cox, S. Ross, M Rubinstein, Option pricing: a simplified approach, J. financial Economics 7 (1979) 229–264.

[6] G. Courtandon, A more accurate finite dieeference approximation for the valuation of options, J. Finantial Quant. Anal. 17 (1982) 697–703.

[7] J.C. Hull, A. White, The use of control variate technique in option pricing, J. Financial Economics Quant. Anal. 23 (1988) 237–251.

[8] S. Ikoene, J. Toivannen, Operator splitting methods for American option pricing, Appl. Math. Lett., 17 (2004), 809–814.

[9] Y. Kwon, Y. Lee, A seond-order finite difference methods for option pricing under jump-diffusion models, (2010) preprint.

[10] Y. Kwon, Y. Lee, A seond-order tridiagonal method for American options under jump-diffusion models, (2010) preprint.

[11] C.V. Loan, Computional Framworks for the Fats Fourier Transform, Frontiers in Applied Mathematics, vol. 10, SIAM, Philadelphia, PA, 1992.

[12] E. Schwartz, The valuation of warrants: implementing a new approach, J. Financial Econimocs 13 (3) (1977) 79–93.

[13] J. Toivanen, Numerical valustion of European and American options under Kou's jump-difusion model, SIAM L. Sci. Comput., 30 (2008), 1949–1970

[14] C. Vazquez, An upwind numerical approach for an American and European option pricing model, Applied Mathematics and Computations 97 (1988) 273–286

[15] X.L. Zang, Numerical analysis of American option pricing in a jump-diffusion model, Math. Oper. Res., 22 (1997), 668–690.

Department of Mathematics, POSTECH, Republic of Korea