Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 11, Number 2, 2009, pages 63–79

ASARC Workshop
Daecheon & Muju 2009

EVALUATION OF THE DEDEKIND ZETA FUNCTIONS AT $s = -1$ OF THE SIMPLEST QUARTIC FIELDS

HYUN KWANG KIM AND JUN HO LEE

Abstract. The simplest quartic field was introduced by M. Gras and studied by A. J. Lazarus. In this paper, we will evaluate the values of the Dedekind zeta functions at $s = -1$ of the simplest quartic fields. We first introduce Siegel's formula for the values of the Dedekind zeta function of a totally real number field at negative odd integers, and will apply Siegel's formula to the simplest quartic fields. In the second, we will develop basic arithmetic properties of the simplest quartic fields which will be necessary in our computation. We will compute the discriminant, ring of integers, and different of the simplest quartic fields. In the third, we will give a full description for a Siegel lattice of the simplest quartic fields, and will develop a method of computing sum of divisor function for ideals. Finally, by combining these results, we compute the values of the Dedekind zeta function at $s = -1$ of the simplest quartic fields.

1. Siegel's formula

Let K be a totally real number field and $\zeta_K(s)$ the Dedekind zeta function of K. Siegel[7] developed an ingenious method of evaluating values of the zeta function of K at negative odd integers by using the finite dimensionality of elliptic modular forms. However, evaluation of the values of the zeta function by means of Siegel's formula requires complicated computations in algebraic number theory, since the formula involves terminology of algebraic number theory such as norm, trace, and different of K. The problem of expressing zeta values in terms of elementary functions was first studied by Zagier[8]. Siegel's formula has been exploited by Zagier to give an elementary expression for $\zeta_K(1-2b)$, where K is a real quadratic field and b is a positive integer, which involves only rational integers and not algebraic

2000 *Mathematics Subject Classification.* Primary 11R42; Secondary 11R16.

numbers or norm of ideals. In [4](resp. [1]), the authors computed the values of the zeta function of the simplest cubic fields(resp. non-normal totally real cubic fields) by using Siegel's formula. In this paper, we will be interested in evaluating zeta values of a certain class of totally real quartic fields, which are called the simplest quartic fields. We first introduce Siegel's formula for the values of the Dedekind zeta function of a totally real algebraic number field at negative odd integers.

Let K be an algebraic number field and $\mathcal{O}_K$ the ring of integers of K. For an ideal I of $\mathcal{O}_K$, we define the sum of ideal divisors function $\sigma_r(I)$ by

$$\sigma_r(I) = \sum_{J|I} N_{K/\mathbb{Q}}(J)^r, \tag{1}$$

where J runs over all ideals of $\mathcal{O}_K$ which divide I. Note that, if $K = \mathbb{Q}$ and $I = (n)$, our definition coincides with the usual sum of divisors function

$$\sigma_r(n) = \sum_{\substack{d|n \\ d>0}} d^r. \tag{2}$$

Now let K be a totally real algebraic number field. For $l, s = 1, 2, \ldots$, we define

$$S_l^K(2b) = \sum_{\substack{\nu \in \delta_K^{-1} \\ \nu \gg 0 \\ \mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l}} \sigma_{2b-1}((\nu)\delta_K), \tag{3}$$

where ν runs over all totally positive elements in the inverse of the different of K with a given trace l. Later we shall study the sum (3) intensively. At this moment, we remark that this is a finite sum. We now state Siegel's formula.

Theorem 1.1. (*Siegel* [7]) *Let b be a natural number, K a totally real algebraic number field of degree n, and $h = 2bn$. Then*

$$\zeta_K(1-2b) = 2^n \sum_{l=1}^{r} b_l(h) S_l^K(2b). \tag{4}$$

The numbers $r \geq 1$ and $b_1(h), \ldots, b_r(h) \in \mathbb{Q}$ depend only on h. In particular,

$$r = \dim_{\mathbb{C}} \mathcal{M}_h, \tag{5}$$

where $\mathcal{M}_h$ denotes the space of modular forms of weight h. Thus by a well-known formula,

$$r = \begin{cases} [\frac{h}{12}] & \textit{if} \quad h \equiv 2 \pmod{12} \\ [\frac{h}{12}] + 1 & \textit{if} \quad h \not\equiv 2 \pmod{12}. \end{cases}$$

Proof. See [7] or [8] □

D. Zagier[8] computed the values of $b_l(h)$ for $4 \le h \le 40$, and we obtain:

Corollary 1.2. *Let K be a totally real quartic number field. Then*

$$\zeta_K(-1) = 2^4 \cdot \frac{1}{480} \cdot S_1^K(2). \tag{6}$$

2. The simplest quartic fields

Let K be a quartic field defined by the polynomial over $\mathbb{Q}$

$$P_t(X) = X^4 - tX^3 - 6X^2 + tX + 1,$$

where t is a natural number such that $t^2 + 16$ is not divisible by an odd square. By [5], it is known that

(i) $P_t(X)$ is irreducible over $\mathbb{Q}$, and
(ii) if ϵ is a root of $P_t(X)$, then so is $\frac{\epsilon-1}{\epsilon+1}$.

Since the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

has order 4 in $\mathrm{PGL}(2, \mathbb{Q})$, the field K is a totally real cyclic quartic field. K is called the simplest quartic field. Let G be the Galois group of K and $\alpha = \alpha_1$ be the largest root of $P_t(X)$. Then G is generated by σ, where $\sigma(\alpha) = \frac{\alpha-1}{\alpha+1}$. Put

$$\alpha_2 = \sigma(\alpha) = \frac{\alpha - 1}{\alpha + 1},\ \alpha_3 = \sigma^2(\alpha) = -\frac{1}{\alpha},\ \alpha_4 = \sigma^3(\alpha) = \frac{1 + \alpha}{1 - \alpha}.$$

By a simple root test, we easily get, for $t \ge 5$, that

$$-2 < \alpha_4 < -1 < \alpha_3 < 0 < \alpha_2 < 1 < t < \alpha_1 < t + 1.$$

Furthermore, we estimate that

$$\alpha_1 \cong t,\ \alpha_2 \cong 1,\ \alpha_3 \cong 0,\ \text{and } \alpha_4 \cong -1 \text{ for sufficiently large } t.$$

Let k be the unique quadratic field of $K/\mathbb{Q}$, that is, the fixed field of the subgroup $\langle \sigma^2 \rangle$ of G. Since $\beta = \alpha - \frac{1}{\alpha} = \frac{t+\sqrt{t^2+16}}{2}$ is invariant under σ^2, we have

$$k = \mathbb{Q}(\alpha - \frac{1}{\alpha}) = \mathbb{Q}(\sqrt{t^2 + 16}).$$

Let $\mathcal{O}_k$ be the ring of integers of k, and D_K, $\mathcal{O}_K$, and δ_K be the discriminant, ring of integers, and different of K, respectively. Lazarus[5] computed the discriminant of K. We also need to compute the ring of integers and different of K. We first compute the ring of integers.

Proposition 2.1. *Let K be the simplest quartic field corresponding to an integer t. Let $\mathcal{O}_K$ be the ring of integers of K. Then*

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha^2 \oplus \mathbb{Z}\frac{1+\alpha^3}{2} & \text{if} \quad t \equiv 1 \pmod 2 \\ \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\frac{1+\alpha^2}{2} \oplus \mathbb{Z}\frac{\alpha+\alpha^3}{2} & \text{if} \quad t \equiv 2 \pmod 4 \\ \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\frac{1+\alpha^2}{2} \oplus \mathbb{Z}\frac{1+\alpha+\alpha^2+\alpha^3}{4} & \text{if} \quad t \equiv 4 \pmod 8 \\ \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\frac{1+2\alpha-\alpha^2}{4} \oplus \mathbb{Z}\frac{1+\alpha+\alpha^2+\alpha^3}{4} & \text{if} \quad t \equiv 0 \pmod 8. \end{cases}$$

To compute the different δ_K of K, we borrow a theorem from [9].

Theorem 2.2. *Let $K = \mathbb{Q}(u)$ be a finite extension of $\mathbb{Q}$ and $F(X)$ be the minimal polynomial of u over $\mathbb{Q}$ and $R = \mathbb{Z}[u]$. Then $\mathcal{O}_K F'(u) = \mathcal{C}\mathcal{D}$, where $\mathcal{C}$ is the conductor of $\mathcal{O}_K$ in R and $\mathcal{D}$ is the different of $\mathcal{O}_K$.*

We now compute the different δ_K of K.

Theorem 2.3. *Let K be the simplest quartic field corresponding to an integer t and let δ_K be the different of K. Then*

$$\delta_K = \begin{cases} \frac{P_t'(\alpha)}{1+\alpha}\mathcal{O}_K & \text{if} \quad t \equiv 1 \pmod 2 \\ \frac{P_t'(\alpha)}{2}\mathcal{O}_K & \text{if} \quad t \equiv 2 \pmod 4 \\ \frac{P_t'(\alpha)}{2(1+\alpha)}\mathcal{O}_K & \text{if} \quad t \equiv 4 \pmod 8 \\ \frac{P_t'(\alpha)}{4}\mathcal{O}_K & \text{if} \quad t \equiv 0 \pmod 8. \end{cases}$$

We summarize the proceeding results in the following table.

	D_K	$\mathcal{O}_k$	$\mathcal{O}_K$	δ_K
$t \equiv 1 \pmod 2$	$(t^2+16)^3$	$\mathbb{Z}[\beta]$	$1, \alpha, \alpha^2, \frac{1+\alpha^3}{2}$	$\frac{P_t'(\alpha)}{1+\alpha}\mathcal{O}_K$
$t \equiv 2 \pmod 4$	$\frac{1}{4}(t^2+16)^3$	$\mathbb{Z}[\frac{1}{2}\beta]$	$1, \alpha, \frac{1+\alpha^2}{2}, \frac{\alpha+\alpha^3}{2}$	$\frac{P_t'(\alpha)}{2}\mathcal{O}_K$
$t \equiv 4 \pmod 8$	$\frac{1}{16}(t^2+16)^3$	$\mathbb{Z}[\frac{1}{2}\beta]$	$1, \alpha, \frac{1+\alpha^2}{2}, \frac{1+\alpha+\alpha^2+\alpha^3}{4}$	$\frac{P_t'(\alpha)}{2(1+\alpha)}\mathcal{O}_K$
$t \equiv 0 \pmod 8$	$\frac{1}{64}(t^2+16)^3$	$\mathbb{Z}[\frac{1}{2}+\frac{\beta}{4}]$	$1, \alpha, \frac{1+2\alpha-\alpha^2}{4}, \frac{1+\alpha+\alpha^2+\alpha^3}{4}$	$\frac{P_t'(\alpha)}{4}\mathcal{O}_K$

3. Description of a Siegel lattice

In this section, we first discuss what is needed for the computation of zeta values by means of Siegel's formula. Next, we introduce the notion of a Siegel lattice which will be important in our computation. Finally, we shall give a full description of a Siegel lattice for the simplest quartic fields. As a result we will derive a formula for the number of points in a Siegel lattice.

The essence of Siegel's formula is that it transforms an infinite series(i.e., the value of the zeta function) into a finite sum involving $S_l^K(2b)$s, where $S_l^K(2b)$ itself is a finite sum of powers of ideal divisors of an integral ideal $(\nu)\delta_K$ where ν runs over the elements in K which satisfy the conditions described in (3). Therefore the description of ν's in K which satisfy the conditions in (3) is of crucial importance in our computation. This can be accomplished by the notion of a Siegel lattice which was first introduced in [3].

Let K be a totally real algebraic number field of degree n and S_K(or simply S) be the set of elements in K which satisfy the Siegel's conditions described in (3). Fix an integral basis $\alpha_1, \ldots, \alpha_n$ of K. For $\nu \in K$, we can write

$$\nu = x_1\alpha_1 + \cdots + x_n\alpha_n, \tag{7}$$

where $x_i \in \mathbb{Q}$. Then we have an embedding $\phi\colon\ K \longrightarrow \mathbb{R}^n$ given by

$$\phi(\nu) = (x_1, \ldots, x_n). \tag{8}$$

The condition $\nu \in \delta_K^{-1}$ implies that the denominator of x_i, $i = 1, 2, \ldots, n$, is bounded by D_K, where D_K denotes the discriminant of K. The condition $\mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l$ is equivalent to saying that $\phi(\nu)$ lies in the hyperplane

$$a_1x_1 + \cdots + a_nx_n = l, \tag{9}$$

where $a_i = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i)$. Finally the condition $\nu \gg 0$ becomes n distinct linear inequalities defined over K in the variables $(x_1, \ldots, x_n)$. Therefore the elements in S_K can be put into one-to-one correspondence with the lattice points in a bounded $(n-1)$-dimensional region under ϕ. We shall call this set(or any set which can be put into one-to-one correspondence with this set under a suitable linear transformation) as a Siegel lattice for K and denote it by T_K(or simply T). Notice that equation (3) expresses $S_l^K(2b)$ as a weighted sum of ideal divisor functions over a Siegel lattice. Hence the description of a Siegel lattice is very important in the computation of $S_l^K(2b)$. First, we consider for the case $t \equiv 1 \pmod 2$.

3.1. $t \equiv 1 \pmod 2$**.** Let K be the simplest quartic field corresponding to an odd integer t. Recall that the discriminant D_K, ring of integers $\mathcal{O}_K$, and different δ_K of K are given respectively by

$$D_K = (t^2 + 16)^3,$$

$$\mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\alpha^2 \oplus \mathbb{Z}\frac{1+\alpha^3}{2},$$

$$\delta_K = \frac{P_t'(\alpha)}{1+\alpha}\mathcal{O}_K = (p + q\alpha + r\alpha^2 + s\alpha^3)\mathcal{O}_K,$$

where $p = \frac{t+4}{2}$, $q = -\frac{t^2-3t+28}{2}$, $r = \frac{t^2-3t+4}{2}$, $s = -\frac{t-4}{2}$.
Let ν be an element of K. We can write

$$\nu = e + f\alpha + g\alpha^2 + h\alpha^3, \quad e, f, g, h \in \mathbb{Q}. \tag{10}$$

Now suppose that ν satisfies the Siegel's conditions, i.e.,

$$\nu \in \delta_K^{-1},\ \mathrm{Tr}(\nu) = l,\ \nu \gg 0. \tag{11}$$

1. $\nu \in \delta_K^{-1}$

Let $\nu = e + f\alpha + g\alpha^2 + h\alpha^3$, $e, f, g, h \in \mathbb{Q}$, be an element of δ_K^{-1}. Since $\nu \in \delta_K^{-1}$ if and only if $\nu(\frac{t+4}{2} - \frac{t^2-3t+28}{2}\alpha + \frac{t^2-3t+4}{2}\alpha^2 - \frac{t-4}{2}\alpha^3) \in \mathcal{O}_K$, we can write

$$\nu(\frac{t+4}{2} - \frac{t^2-3t+28}{2}\alpha + \frac{t^2-3t+4}{2}\alpha^2 - \frac{t-4}{2}\alpha^3) = \frac{u}{2} + v\alpha + w\alpha^2 + \frac{z}{2}\alpha^3,$$

where $u, v, w, z \in \mathbb{Z}$ and $u \equiv z \pmod 2$. From comparison of coefficient of $\alpha^i (0 \le i \le 3)$, we obtain the following equation:

$$\begin{bmatrix} \frac{t+4}{2} & \frac{t-4}{2} & -\frac{t+4}{2} & -\frac{t-4}{2} \\ -\frac{t^2-3t+28}{2} & \frac{t^2-3t+4}{2} & -\frac{t^2+3t+4}{2} & -\frac{t^2-3t+4}{2} \\ \frac{t^2-3t+4}{2} & -\frac{t^2+3t+4}{2} & \frac{t^2+3t+28}{2} & -\frac{t^2-3t+28}{2} \\ -\frac{t-4}{2} & \frac{t+4}{2} & \frac{t-4}{2} & \frac{2t^2-t+24}{2} \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} \frac{u}{2} \\ v \\ w \\ \frac{z}{2} \end{bmatrix}. \tag{12}$$

By solving the system (12), we have

$$\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \frac{1}{2D} \begin{bmatrix} 2t-3 & -3 & -1 & -1 \\ -t-17 & -t-3 & -t-3 & -t-1 \\ -3t+1 & -t+1 & -t+3 & -t+3 \\ 3 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{u}{2} \\ v \\ w \\ \frac{z}{2} \end{bmatrix}, \tag{13}$$

where $D = t^2 + 16$. By an easy calculation, we can write

$$e = \frac{a}{2D},\ f = \frac{b}{D},\ g = \frac{c}{D},\ h = \frac{d}{2D}, \tag{14}$$

where a, b, c, d are integers. By plugging (14) into (12) and adding the first and fourth row, we obtain

$$ta - 8b - 2tc - (t^2 + 12)d \equiv 0 \pmod{2D}.$$

Since t is odd,

$$a \equiv d \pmod 2. \tag{15}$$

Hence, we conclude that

$$\nu = e + f\alpha + g\alpha^2 + h\alpha^3 = \frac{a}{2D} + \frac{b}{D}\alpha + \frac{c}{D}\alpha^2 + \frac{d}{2D}\alpha^3, \tag{16}$$

where a, b, c, d are integers such that $a \equiv d \pmod 2$.

2. $\mathrm{Tr}(\nu) = l$

If $\mathrm{Tr}(\nu) = l$, then

$$2a + tb + (t^2 + 12)c + \frac{t^3 + 15t}{2}d = Dl. \tag{17}$$

From (17),

$$b = tl - 2(\frac{a + 6c - 8l}{t}) - tc - \frac{t^2 + 15}{2}d. \tag{18}$$

Since t is odd,

$$a + 6c - 8l \equiv 0 \pmod t. \tag{19}$$

Now we introduce a new variable s by the formula

$$s = \frac{a + 6c - 8l}{t}. \tag{20}$$

By substitution of (20) into (18), we have

$$b = tl - 2s - tc - \frac{t^2 + 15}{2}d.$$

Conversely, if a, b, c, d are integers satisfying (16) and (17), then $\nu = \frac{a}{2D} + \frac{b}{D}\alpha + \frac{c}{D}\alpha^2 + \frac{d}{2D}\alpha^3 \in \delta_K^{-1}$ and $\mathrm{Tr}(\nu) = l$.

3. $\nu \gg 0$

Let

$$S = \{\nu | \nu \in \delta_K^{-1},\ \mathrm{Tr}(\nu) = l,\ \nu \gg 0\}.$$

Define

$$\mathcal{S} : S \longrightarrow \mathbb{R}^3$$

by

$$\mathcal{S}(\nu) = (s, c, d).$$

Note that the condition $\nu \gg 0$ is equivalent to the fact that

$$2D\nu = a + 2b\alpha + 2c\alpha^2 + d\alpha^3 \gg 0,$$

where $a \equiv d \pmod 2$. By (17), (18), (20), this condition gives four linear inequalities in the variables s, c, d defined over K, namely

$$P_i = (t-4\alpha_i)s + 2(\alpha_i^2 - t\alpha_i - 3)c + (\alpha_i^3 - (t^2+15)\alpha_i)d + (8+2t\alpha_i)l > 0 \ (1 \le i \le 4),$$

where $s, c, d \in \mathbb{Z}$ and $s \equiv d \pmod 2$.

To find all νs which satisfy the Siegel's conditions is equivalent to find lattice points (s, c, d) which lie in the interior of the tetrahedron determined by $P_i(1 \le i \le 4)$ and satisfy $s \equiv d \pmod 2$. Since the tetrahedron is not of 'good' shape to visualize their regularity and symmetry of lattice points, we shall use a subsidiary transformation T. Define

$$U : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

by

$$U(s, c, d) = (\frac{s+d}{2}, \frac{(t+1)s - 4c + (t+1)d}{4}, \frac{s+3d}{2}) = (x, y, z).$$

Put

$$T = U \circ \mathcal{S}.$$

By the transformation T, to find all νs which satisfy the Siegel's conditions is equivalent to find lattice points (x, y, z) which lie in the interior of a tetrahedron determined by $P_i(1 \le i \le 4)$, where

$$P_i(x,y,z) = (-\alpha_i^3 + (t+1)\alpha_i^2 + (3-t)\alpha_i - 3)x - 2(\alpha_i^2 - t\alpha_i - 3)y$$
$$+(\alpha_i^3 - (t^2+11)\alpha_i - t)z + (2t\alpha_i + 8)l) > 0.$$

For $1 \le i \le 4$ and $\{j, k, m\} = \{1, 2, 3, 4\} \setminus \{i\}$, a long and tedious computation yields that

$$-\frac{l}{\alpha_i} \text{ is a z-coordinate of the point of intersection of } P_j = P_k = P_m = 0.$$

We summarize the above computation in the following proposition.

Proposition 3.1. *Let S be the set of elements in K which satisfy the Siegel's conditions described in* (3) *and T be the set of lattice points in the interior of the tetrahedron determined by $P_i(1 \le i \le 4)$. For $\nu \in S$, we can write*

$$\nu = \frac{a}{2D} + \frac{b}{D}\alpha + \frac{c}{D}\alpha^2 + \frac{d}{2D}\alpha^3, \text{ where } a, b, c, d \in \mathbb{Z}. \tag{21}$$

Then the mapping $\eta : S \longrightarrow T$ given by $\eta(\nu) = (x, y, z)$, where

$$x = \frac{1}{2t}(a + 6c + td - 8l),\ y = \frac{1}{4t}\{(t+1)a + 2(t+3)c + t(t+1)d - 8(t+1)l\},$$

$$\text{and } \ z = \frac{1}{2t}(a + 6c + 3td - 8l),$$

gives a one-to-one correspondence between S *and* T*. The inverse mapping* $\tau : T \longrightarrow S$ *is given by*

$$\tau(x, y, z) = \nu = \frac{a}{2D} + \frac{b}{D}\alpha + \frac{c}{D}\alpha^2 + \frac{d}{2D}\alpha^3,$$

where

$$a = -3x + 6y - tz + 8l, \ b = \frac{1}{2}(3 - t) + ty - \frac{1}{2}(t^2 + 11)z + tl,$$
$$c = \frac{1}{2}(t + 1)x - y, \ d = z - x.$$

We now describe Galois action on a Siegel lattice. We start with the following simple observation.

Lemma 3.2. *Let* K *be a totally real Galois extension of* $\mathbb{Q}$ *with Galois group* G*. If* $\nu \in K$ *satisfies the Siegel's conditions described in* (3)*, then so does* $\sigma(\nu)$ *for any* $\sigma \in G$*.*

By Proposition 3.1 and Lemma 3.2, the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ acts on the set S and S can be put into one-to-one correspondence with a Siegel lattice T. Therefore, we have the induced Galois action on T. Now we return to the simplest quartic field case and describe Galois action T.

Proposition 3.3. (*Galois action on a Siegel lattice*) *Let* ν *satisfy the Siegel's conditions and let* (x, y, z) *be the points corresponding to* ν *by* η *in Proposition* 3.1*. Then*

$$\eta(\sigma(\nu)) = (x_1, y_1, z_1),$$

where

$$x_1 = -2x + y - \frac{1}{2}(t + 3)z + tl,$$
$$y_1 = \frac{1}{2}(t - 1)x - \frac{1}{2}(t - 1)y + \frac{1}{4}(t^2 + 3)z,$$
$$z_1 = x - y + \frac{1}{2}(t + 1)z.$$

Remark 3.4. *By a successive application of Proposition* 3.3*, we have*

$$\eta(\sigma^2(\nu)) = (x_2, y_2, z_2),$$

where

$$x_2 = 2x + 3z - tl,$$
$$y_2 = -\frac{1}{2}(t - 3)x - y - \frac{1}{2}(t - 3)z + \frac{1}{2}(t^2 - t)l,$$
$$z_2 = -x - 2z + tl,$$

and

$$\eta(\sigma^3(\nu)) = (x_3, y_3, z_3),$$

where

$$x_3 = -x - y + \frac{1}{2}(t-3)z + tl,$$
$$y_3 = -x + \frac{1}{2}(t-1)y - \frac{1}{4}(t^2 - 2t + 9)z + tl,$$
$$z_3 = y - \frac{1}{2}(t-1).$$

Proposition 3.5. *Let G act on S as in Proposition 3.3. If $\mathrm{Tr}(\nu) = l$ is odd, then every G-orbit contains 4 points. In particular, N_l is divisible by 4, where N_l denotes the number of lattice points in T which corresponds to $\mathrm{Tr}(\nu) = l$.*

For $(x, y, z) \in T$, by Proposition 3.1, the corresponding ν is given by

$$\nu = \frac{a}{2D} + \frac{b}{D}\alpha + \frac{c}{D}\alpha^2 + \frac{d}{2D}\alpha^3,$$

where

$$a = -3x + 6y - tz + 8l,\ b = \frac{1}{2}(3-t) + ty - \frac{1}{2}(t^2 + 11)z + tl,$$
$$c = \frac{1}{2}(t+1)x - y,\ d = z - x.$$

Recall that

$$\delta_K = (\frac{t+4}{2} - \frac{t^2 - 3t + 28}{2}\alpha + \frac{t^2 - 3t + 4}{2}\alpha^2 - \frac{t-4}{2}\alpha^3)\mathcal{O}_K.$$

Then we have

$$(\nu)\delta_K = (s + u\alpha + v\alpha^2 + w\alpha^3),$$

where

$$s = -\frac{x}{2} + \frac{y}{2} - \frac{(t-3)z}{4} + \frac{l}{2},\ u = -\frac{(t-3)x}{2} + \frac{(t-5)y}{2} - \frac{(t^2 - 4t + 3)z}{4} + \frac{(t-7)l}{2},$$
$$v = \frac{(t+1)x}{2} - \frac{(t+1)y}{2} + \frac{(t^2 + 2t + 1)z}{4} - \frac{(t-1)l}{2},\ w = -\frac{x}{2} + \frac{y}{2} - \frac{(t+1)z}{4} - \frac{l}{2}.$$

Let $N(x, y, z; l)$ denote the norm function $N_{K/\mathbb{Q}}(s + u\alpha + v\alpha^2 + w\alpha^3)$. By an elementary computation, we obtain

$$N(x, y, z; l) = f^2 + tfg - 4g^2, \tag{22}$$

where $f = s^2 - u^2 + v^2 - w^2 + 6(sv - uw) + 4tsw$, $g = su + uv + vw + t(sv - uw) + (t^2 + 7)sw$.

The function $N(x, y, z; l)$ will be useful in description of T. Note that T is the set of lattice points in (x, y, z)-plane which lies inside of the tetrahedron surrounded by the planes $P_1 = 0, P_2 = 0, P_3 = 0$, and $P_4 = 0$; $N(x, y, z; l) < 0$ if a point (x, y, z) lies inside the tetrahedron.

Note that each point on the boundary of a tetrahedron moves to a point on another boundary by Galois action. If we can find lattice points near to appropriate boundary, then we can find Siegel lattice points by observing movements of conjugates of the points. We observe that P_3-plane is approximately arranged along xy-plane. We choose P_3-plane and find lattice points near to P_3-plane. Next, if we observe the movement of conjugates of Siegel lattice points in $z = 0$, then we can seek for all Siegel lattice points for $z = k$. Combining these data, we derive a formula for the number of points in a Siegel lattice.

Theorem 3.6. *Let t be an odd natural number such that $t^2 + 16$ is not divisible by odd square and K be the simplest quartic field corresponding to t. Let N denote the number of a Siegel lattice for K which corresponds to* $\mathrm{Tr}(\nu) = 1$. *Then we have*

$$N = \begin{cases} 12 & \text{if} \quad t = 1 \\ \frac{t^3+23t+24}{6} & \text{if} \quad t \geq 5. \end{cases}$$

3.2. $t \equiv 2 \pmod 4$**.** Recall that the discriminant D_K, ring of integers $\mathcal{O}_K$, and different δ_K of K are given respectively by

$$D_K = \frac{1}{4}(t^2+16)^3,$$

$$\mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\frac{1+\alpha^2}{2} \oplus \mathbb{Z}\frac{\alpha+\alpha^3}{2},$$

$$\delta_K = \frac{P_t'(\alpha)}{2}\mathcal{O}_K = (\frac{t}{2} - 6\alpha - \frac{3t}{2}\alpha^2 + 2\alpha^3)\mathcal{O}_K.$$

We will compute directly δ_K^{-1}.

Letting $D = t^2 + 16$, by a direct calculation, we obtain

$$\begin{aligned} \delta_K^{-1} &= \frac{1}{D}\{(2t - 17\alpha - 3t\alpha^2 + 3\alpha^3)\mathbb{Z} \oplus (-3 - t\alpha + \alpha^2)\mathbb{Z} \\ &\quad \oplus (t - 10\alpha - 2t\alpha^2 + 2\alpha^3)\mathbb{Z} \oplus (-2 - t\alpha + 2\alpha^2)\mathbb{Z}\} \\ &= \{\frac{1}{D}((2at - 3b + ct - 2d) + (-17a - bt - 10c - dt)\alpha \\ &\quad +(-3at + b - 2ct + 2d)\alpha^2 + (3a + 2c)\alpha^3) \mid a, b, c, d \in \mathbb{Z}\}. \end{aligned} \tag{23}$$

If $\nu \in \delta_K^{-1}$ and $\mathrm{Tr}(\nu) = l$, then we have $d = l$. Define

$$\mathcal{S} : S \longrightarrow \mathbb{R}^3$$

by

$$S(\nu) = (a, b, c + tl) = (x, y, z).$$

The condition that $\nu \gg 0$ gives four linear inequalities in the variables x, y, z defined over K:

$$P_i(x,y,z) = (2t - 17\alpha_i - 3t\alpha_i^2 + 3\alpha_i^3)x + (-3 - t\alpha_i + \alpha_i^2)y$$
$$+(t - 10\alpha_i - 2t\alpha_i^2 + 2\alpha_i^3)z + (-(t^2+2) + 9t\alpha_i + 2(t^2+1)\alpha_i^2 - 2t\alpha_i^3)l > 0$$
$$(1 \le i \le 4).$$

For $1 \le i \le 4$ and $\{j,k,m\} = \{1,2,3,4\} \setminus \{i\}$, We obtain

$\alpha_i l$ is a z-coordinate of the point of intersection of $P_j = P_k = P_m = 0$.

We summarize the above computation in the following proposition.

Proposition 3.7. *Let S be the set of elements in K which satisfy the Siegel's conditions described in* (3) *and T be the set of lattice points in the interior of the tetrahedron determined by $P_i (1 \le i \le 4)$. For $\nu \in S$, by* (23) *we can write*

$$\nu = \frac{e}{D} + \frac{f}{D}\alpha + \frac{g}{D}\alpha^2 + \frac{h}{D}\alpha^3, \tag{24}$$

where

$$e = 2at - 3b + ct - 2d,\ f = -17a - bt - 10c - dt,$$
$$g = -3at + b - 2ct + 2d,\ h = 3a + 2c,\ a,b,c,d \in \mathbb{Z}.$$

Then the mapping $\eta : S \longrightarrow T$ given by $\eta(\nu) = (x,y,z)$, where

$$x = \frac{2(t^2+5)e - 5ft + (t^2+30)g - (3t^2+40)l}{t^3+15t},$$
$$y = \frac{-4e - ft + 3g - (t^2+14)l}{t^2+15},$$
$$z = \frac{-(3t^2+17)e + 7ft - (2t^2+51)g + (t^4+20t^2+68)l}{t^3+15t},$$

gives a one-to-one correspondence between S and T. The inverse mapping $\tau : T \longrightarrow S$ is given by

$$\tau(x,y,z) = \nu = \frac{e}{D} + \frac{f}{D}\alpha + \frac{g}{D}\alpha^2 + \frac{h}{D}\alpha^3,$$

where

$$e = 2tx - 3y + tz - (t^2+2)l,\ f = -17x - ty - 10z + 9tl,$$
$$g = -3tx + y - 2tz + 2(t^2+1)l,\ d = 3x + 2z - 2tl.$$

Proposition 3.8. (*Galois action on a Siegel lattice*) *Let ν satisfy the Siegel's conditions and let (x,y,z) be the points corresponding to ν by η in Proposition* 3.8. *Then*

$$\eta(\sigma(\nu)) = (x_1, y_1, z_1),$$

where

$$x_1 = y + \frac{1}{2}(t+4)l,\ y_1 = -x + \frac{1}{2}(t-4)l,\ z_1 = -x - y - z + (t-2)l.$$

Remark 3.9. *By a successive application of Proposition* 3.9*, we have*

$$\eta(\sigma^2(\nu)) = (x_2, y_2, z_2),$$

where

$$x_2 = -x + tl,\ y_2 = -y - 4l,\ z_2 = 2x + z - tl,$$

and

$$\eta(\sigma^3(\nu)) = (x_3, y_3, z_3),$$

where

$$x_3 = -y + \frac{1}{2}(t-4)l,\ y_3 = x - \frac{1}{2}(t+4)l,\ z_3 = -x + y - z + (t+2)l.$$

Proposition 3.10. *Let G act on S as in Proposition* 3.9*. If* $\mathrm{Tr}(\nu) = 1$*, then the points of the form* $(\frac{t}{2}, -2, z)$ *are fixed by* σ^2*. Every G-orbit having no the points contains* 4 *points.*

For $(x, y, z) \in T$, by Proposition 3.8, the corresponding ν is given by

$$\nu = \frac{e}{D} + \frac{f}{D}\alpha + \frac{g}{D}\alpha^2 + \frac{h}{D}\alpha^3,$$

where

$$e = 2tx - 3y + tz - (t^2+2)l,\ f = -17x - ty - 10z + 9tl,$$
$$g = -3tx + y - 2tz + 2(t^2+1)l,\ d = 3x + 2z - 2tl.$$

Recall that

$$\delta_K = (\frac{t}{2} - 6\alpha - \frac{3t}{2}\alpha^2 + 2\alpha^3)\mathcal{O}_K.$$

Then we have

$$(\nu)\delta_K = (s + u\alpha + v\alpha^2 + w\alpha^3),$$

where

$$s = \frac{1}{2}(2x + z - tl),\ u = y + \frac{1}{2}l,\ v = \frac{1}{2}(z - tl),\ w = \frac{1}{2}l.$$

Let $N(x, y, z; l)$ denote the norm function $N_{K/\mathbb{Q}}(s + u\alpha + v\alpha^2 + w\alpha^3)$. By an elementary computation, we obtain

$$N(x, y, z; l) = f^2 + tfg - 4g^2, \tag{25}$$

where $f = s^2 - u^2 + v^2 - w^2 + 6(sv - uw) + 4tsw$, $g = su + uv + vw + t(sv - uw) + (t^2+7)sw$.

Note that T is the set of lattice points in (x, y, z)-plane which lies inside of the tetrahedron surrounded by the planes $P_1 = 0, P_2 = 0, P_3 = 0$, and $P_4 = 0$;

$N(x, y, z; l) > 0$ if a point (x, y, z) lies inside the tetrahedron. We observe that P_1-plane is approximately arranged along xy-plane. We choose P_1-plane and find lattice points near to P_1-plane. Next, if we observe the movement of conjugates of Siegel lattice points in $z = 0, 1$, then we can seek for all Siegel lattice points for $z = k$. Combining these data, we derive a formula for the number of points in a Siegel lattice.

Theorem 3.11. *Suppose $t \equiv 2 \pmod 4$. Let K be the simplest quartic field corresponding to t. Let N denote the number of a Siegel lattice for K which corresponds to* $\mathrm{Tr}(\nu) = 1$. *Then we have*

$$N = \frac{t^3 + 26t + 60}{12}.$$

We can also give a full description of a Siegel lattice for $t \equiv 4 \pmod 8$ and $t \equiv 0 \pmod 8$ by a similar procedure as a previous method. We state only the result for the number of a Siegel lattice.

Theorem 3.12. *Suppose $t \equiv 4 \pmod 8$. Let K be the simplest quartic field corresponding to t. Let N denote the number of a Siegel lattice for K which corresponds to* $\mathrm{Tr}(\nu) = 1$. *Then we have*

$$N = \frac{t^3 + 20t + 24}{24}.$$

Theorem 3.13. *Suppose $t \equiv 0 \pmod 8$. Let K be the simplest quartic field corresponding to t. Let N denote the number of a Siegel lattice for K which corresponds to* $\mathrm{Tr}(\nu) = 1$. *Then we have*

$$N = \frac{t^3 + 44t}{48}.$$

4. Values of zeta functions

In this section, we will evaluate $\zeta_K(-1)$ where K is the simplest quartic field by combining results in previous chapters. By Corollary 1.2, the computation of $\zeta_K(-1)$ is equivalent to the computation of $S_1^K(2)$. Recall that

$$S_l^K(2b) = \sum_{\substack{\nu \in \delta_K^{-1} \\ \nu \gg 0 \\ \mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l}} \sigma_{2b-1}((\nu)\delta_K).$$

It follows from the unique factorization of ideals into prime ideals that for every $\nu \in \delta_K^{-1}$,

$$\sigma_{2b-1}((\nu)\delta_K) = \sigma_{2b-1}((\nu^{(i)})\delta_K),$$

where $\nu^{(i)} = \sigma^i(\nu)(1 \le i \le 4)$. Hence, it makes the computation of $S_l^K(2b)$ easier if we can choose a set of representatives for Galois action on a Siegel lattice. We omit here the statement for a set of representatives for Galois action.

To calculate the summand $S_l^K(2b)$, we need to know the prime ideal decomposition of $(\nu)\delta_K$. The following lemma will be helpful to classify prime ideal decomposition in K.

Lemma 4.1. *Let p be a rational odd prime. If p is ramified(resp. inert) in $k/\mathbb{Q}$, then it is totally ramified(resp. totally inert) in $K/\mathbb{Q}$.*

By Lemma 4.1, if p is a rational odd prime, there are only four types of prime ideal decomposition in $K/\mathbb{Q}$:

(i) Type 1 : p is totally ramified in $K/\mathbb{Q}$, i.e., $(p) = Q^4$,
(ii) Type 2 : p splits completely in $K/\mathbb{Q}$, i.e., $(p) = Q_1Q_2Q_3Q_4$,
(iii) Type 3 : p is totally inert in $K/\mathbb{Q}$, i.e., (p) remains prime in $\mathcal{O}_K$,
(iv) Type 4 : p splits in $k/\mathbb{Q}$(so $(p) = Q_1Q_2$), and both Q_1, Q_2 are inert in K/k, i.e., $(p) = Q_1Q_2$.

If $p = 2$, there are other types of prime ideal decomposition in $K/\mathbb{Q}$ according to the case of t. The following well known proposition(cf. [6]) will be useful in determining prime ideal decomposition.

Proposition 4.2. *Let K be an algebraic number field and let L/K be an extension of degree n; let A(resp. B) be the ring of integers of K(resp. L). Let $L = K(t)$, with $t \in B$, and let $F \in A[X]$ be the minimal of t over K. If P is a nonzero prime ideal of A and $\overline{K} = A/P$, for each polynomial $H \in A[X]$, let $\overline{H} \in \overline{K}[X]$ be obtained by the canonical homomorphism $A \to A/P = \overline{K}$. We assume that one of the following conditions is satisfied:*

(i) *A is a principal ideal domain.*
(ii) *$B = A[t]$.*

Let $(x_1, x_2, \dots, x_n)$ be an integral basis of K, and $\alpha = \sqrt{\frac{\mathrm{disc}_{K/\mathbb{Q}}(1, t, \dots, t^{n-1})}{\mathrm{disc}_{K/\mathbb{Q}}(x_1, x_2, \dots, x_n)}}$. Let P be a nonzero prime ideal of A, such that, in case (i)*, P does not divide $A\alpha$. Let $\overline{F} = \prod_{i=1}^{g} \overline{G_i}^{e_i}$ where $G_i \in A[X]$, the polynomials $\overline{G_1}, \dots, \overline{G_g}$ are distinct and irreducible over $\overline{K}$, $\deg(\overline{G_i}) = f_i$ for $i = 1, \dots, g$. Then $BP = \prod_{i=1}^{g} \overline{Q_i}^{e_i}$ where $Q_1, \dots, Q_g$ are distinct nonzero prime ideals of B and $[B/Q_i : A/P] = f_i$ for every $i = 1, \dots, g$. Moreover, $Q_i \cap A[t] = A[t]P + A[t]G_i(t)$ for $i = 1, \dots, g$.*

Now, we are ready to investigate the prime ideal decomposition of $(\nu)\delta_K$. Finally, using the norm function $N(x, y, z; l)$, we can compute the prime ideal decomposition of $(\nu)\delta_K$ in K. For example, we consider $N(x, y, z; l)$ has the only prime factor p

which is of Type 2. If $N(x,y,z;l)=p$, then $(\nu)\delta_K=Q$, where Q is a prime ideal of $\mathcal{O}_K$ over p. If $N(x,y,z;l)=p^2$, then we have either $(\nu)\delta_K=Q_1^2$ or $(\nu)\delta_K=Q_1Q_2$, where Q_1,Q_2 are prime ideals of $\mathcal{O}_K$ over p. To determine whether $(\nu)\delta_K=Q_1^2$ or $(\nu)\delta_K=Q_1Q_2$, we compute $\mathfrak{A}\mathfrak{A}^{\sigma}\mathfrak{A}^{\sigma^2}$ where $\mathfrak{A}=(\nu)\delta_K$ and $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$. If p divides $\mathfrak{A}\mathfrak{A}^{\sigma}\mathfrak{A}^{\sigma^2}$, then $(\nu)\delta_K=Q_1Q_2$. If p does not divide $\mathfrak{A}\mathfrak{A}^{\sigma}\mathfrak{A}^{\sigma^2}$, then $(\nu)\delta_K=Q_1^2$. Using this method, we can compute the prime ideal decomposition of $(\nu)\delta_K$ in K where ν runs over all the elements in K which satisfies the Siegel conditions, hence we can compute the value $S_l^K(2b)$ and finally compute the zeta values. Now, we give two examples of the computation of $\zeta_K(-1)$.

Example 4.3. (*The case* $t=1$)
The Siegel lattice points for $t=1$ *and their conjugates by Galois action are given as follows:*

$$\begin{aligned} T &= \{(0,-1,0),(0,0,1),(-1,1,1),(2,1,-1),\\ &\quad (1,0,0),(-1,0,1),(1,1,0),(0,0,0),\\ &\quad (2,0,0),(-3,0,2),(3,2,-1),(-1,-1,0)\}. \end{aligned}$$

$$(0,-1,0)\underset{\rightarrow}{\sigma}(0,0,1)\underset{\rightarrow}{\sigma^2}(-1,1,1)\underset{\rightarrow}{\sigma^3}(2,1,-1);N(0,-1,0;1)=-4,$$
$$(1,0,0)\underset{\rightarrow}{\sigma}(-1,0,1)\underset{\rightarrow}{\sigma^2}(1,1,0)\underset{\rightarrow}{\sigma^3}(0,0,0);N(1,0,0;1)=-13,$$
$$(2,0,0)\underset{\rightarrow}{\sigma}(-3,0,2)\underset{\rightarrow}{\sigma^2}(3,2,-1)\underset{\rightarrow}{\sigma^3}(-1,-1,0);N(2,0,0;1)=-1.$$

Since $\alpha=\sqrt{\frac{\mathrm{disc}_{K/\mathbb{Q}}(1,t,\dots,t^{n-1})}{\mathrm{disc}_{K/\mathbb{Q}}(x_1,x_2,\dots,x_n)}}=2$, *we cannot use Proposition* 4.2 *for a prime* 2. *But we can easily check that* 2 *is Type* 4. *It follows that*

$$\zeta_K(-1)=\frac{1}{30}S_1^K(2)=\frac{1}{30}\cdot 4\cdot(14+1+5)=\frac{8}{3}.$$

Example 4.4. (*The case* $t=6$)
A set of representatives of T *for Galois action is given by:*

$$\begin{aligned} R &= \{(4,-1,0),(5,-2,0),(6,-3,0),(3,-1,0),(4,-2,0),\\ &\quad (5,-3,0),(3,-2,0),(4,-3,0),(3,-2,1),(3,-1,1)\}. \end{aligned}$$

Note that the points $(3,-2,0),(3,-2,1)$ *are fixed by* σ^2.

$$\begin{aligned} &N(4,-1,0;1)=61,\ N(5,-2,0;1)=29,\ N(6,-3,0;1)=1,\\ &N(3,-1,0;1)=1,\ N(4,-2,0;1)=53,\ N(5,-3,0;1)=3^2,\\ &N(3,-2,0;1)=13,\ N(4,-3,0;1)=3^2,\ N(3,-2,1;1)=3^2\cdot 13,\\ &N(3,-1,1;1)=2^2\cdot 23. \end{aligned}$$

t	D	$30\zeta_K(-1)$	t	D	$30\zeta_K(-1)$
1	17	$2^4\cdot 5$	12	$2^5\cdot 5$	$2^2\cdot 5\cdot 7\cdot 269$
2	$2^2\cdot 5$	$2^2\cdot 5$	13	$5\cdot 37$	$2^6\cdot 5^5\cdot 19$
4	2^5	5^2	14	$2^2\cdot 53$	$2^2\cdot 5\cdot 7\cdot 37\cdot 173$
5	41	$2^6\cdot 5\cdot 13$	15	241	$2^4\cdot 5\cdot 71\cdot 2137$
6	$2^2\cdot 13$	$2^2\cdot 5\cdot 73$	16	$2^4\cdot 17$	$2^4\cdot 5\cdot 13\cdot 61$
7	$5\cdot 13$	$2^8\cdot 5\cdot 29$	17	$5\cdot 61$	$2^7\cdot 5^2\cdot 41\cdot 281$
8	$2^4\cdot 5$	$2^2\cdot 41$	18	$2^2\cdot 5\cdot 17$	$2^4\cdot 3^2\cdot 5\cdot 29\cdot 337$
9	97	$2^4\cdot 5\cdot 17\cdot 149$	19	$13\cdot 29$	$2^7\cdot 5^3\cdot 53\cdot 109$
10	$2^2\cdot 29$	$2^2\cdot 3\cdot 5^2\cdot 173$	20	$2^5\cdot 13$	$2^2\cdot 5^3\cdot 5101$
11	137	$2^7\cdot 3\cdot 5^2\cdot 97$	21	457	$2^6\cdot 3^2\cdot 5^3\cdot 17\cdot 197$

TABLE 1. Values of $\zeta_K(-1)$ for the first twenty simplest quartic fields.

We can check that 2 *is inert(resp. ramified) in* $k/\mathbb{Q}$*(resp.* K/k*).*
Since $\alpha = \sqrt{\frac{\mathrm{disc}_{K/\mathbb{Q}}(1,t,\ldots,t^{n-1})}{\mathrm{disc}_{K/\mathbb{Q}}(x_1,x_2,\ldots,x_n)}} = 4$*, we can apply Proposition* 4.2 *for a prime* 3*. Note that*

$$x^4 - tx^3 - 6x^2 + tx + 1 \equiv (x^2+x+2)(x^2+2x+2)(\mathrm{mod}\,3).$$

Hence, 3 *is Type* 4*. It follows that*

$$\zeta_K(-1) = \frac{1}{30}S_1^K(2) = \frac{1}{30}\cdot(4\cdot 288 + 2\cdot 154) = \frac{146}{3}.$$

We list the values of $\zeta_K(-1)$ for the first twenty simplest quartic fields(Table 4.1).

REFERENCES

[1] S. J. Cheon, H. K. Kim, and J. H. Lee, *Evaluation of the Dedekind zeta functions of some non-normal totally real cubic fields at negative odd integers*, Manuscripta Math. **124** (2007), 551-560.

[2] Marie-Nicole Gras, *Table Numérique du Nombre de Classes et des Unités des Extensions Cycliques de Degré* 4 *de* $\mathbb{Q}$, Publ. math. fasc. **2**, Fac. Sci. Besançon, 1977/1978.

[3] H. K. Kim and H. J. Hwang, *Values of zeta functions and class number 1 criterion for the simplest cubic fields*, Najoya Math. J. **160** (2000), 161-180.

[4] H. K. Kim and J. S. Kim, *Evaluation of zeta function of the simplest cubic field at negative odd integers*, Math. Comp. **71** (2002), 1243-1262.

[5] A. J. Lazarus, *The class number and unit index of the simplest quartic fields*, Nagoya Math. J. **121** (1991), 1-13.

[6] P. Ribenboim, *Classical Theory of Algebraic Numbers*, Universitext, Springer-Verlag, New York, 2001.

[7] C. L. Siegel, *Berechnung von Zetafuncktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1969** (1969), 87-102.

[8] D. B. Zagier, *On the values at negative integers of the zeta function of a real quadratic field*, Enseignement Math. **22** (1976), 55-95.

[9] O. Zariski, and P. Samuel, *Commutative Algebra*, Springer-Verlag, New York, 1975.

Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea

E-mail address: hkkim@postech.ac.kr

Department of Mathematics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea

E-mail address: joonho2@postech.ac.kr