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CALIBER NUMBER OF REAL QUADRATIC FIELDS

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1. Introduction

Let d be a positive square free integer and D be a discriminant of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. Denote by $[A, B, C]$ a binary quadratic form $Q(X, Y) = AX^2 + BXY + CY^2 \in \mathbb{Z}[X, Y]$. Then $GL_2(\mathbb{Z})$ acts on the set $\mathfrak{Q}(D)$ of primitive binary quadratic forms $[A, B, C]$ of discriminant $D = B^2 - 4AC$ by $S \circ Q(X, Y) = Q(aX + bY, cX + dY)$ for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ and $Q(X, Y) \in \mathfrak{Q}(D)$. Let $H(D)$ be the set of equivalent classes $\mathfrak{Q}(D)/GL_2(\mathbb{Z})$. The cardinality of $H(D)$ is the class number $h(D) = h(d)$ of $\mathbb{Q}(\sqrt{d})$.

A quadratic form $[A, B, C]$ of discriminant D is called *reduced* if the coefficients satisfy the following inequalities:

$$(1) \qquad A > 0,\ B < 0,\ C < 0,\ |B| < \sqrt{D},\ \sqrt{D} - |B| < 2A < \sqrt{D} + |B|.$$

Let $\mathfrak{Q}_{red}(D)$ be the set of reduced forms. The caliber number $\kappa(D) = \kappa(d)$ of $\mathbb{Q}(\sqrt{d})$ is the cardinality of $\mathfrak{Q}_{red}(D)$.

For a real quadratic irrationality w in K, its caliber $m(w)$ is simply the length of the periodic part in the continued fraction expansion. Let ω_Q be a root of $Q(X, 1)$, then $m([Q])$ is actually a class invariant in such a way that $m([Q]) = m(w_{Q_1}) = m(w_{Q_2})$ if Q_1 and Q_2 are in the same class $[Q] \in H(D)$. And it is well-known that $m([Q])$ is the number of reduced forms in the class $[Q]$. Thus the caliber number

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$\kappa(D)$ is rewritten as follows:

$$\kappa(D) = \sum_{[Q]\in H[D]} m([Q])$$

In [3], Lachaud obtained an effective lower bound of $\kappa(d)$ assuming $\zeta_K(\frac{1}{2}) \leq 0$:

$$\kappa(d) > \frac{1}{8.46}\log(d-3). \tag{2}$$

Moreover, Lachaud determined all real quadratic fields with caliber number 1 with assumption of $\zeta_K(\frac{1}{2}) \leq 0$.

Now we explain our works. We find the lower bound of caliber number using splitting prime p in K.

Theorem 1.1. *For a splitting prime p in K, we have*

$$\kappa(d) > 2\Big[\frac{\log\frac{\sqrt{D}}{2}}{\log p}\Big].$$

And we obtain the following theorems

Theorem 1.2. *If $\kappa(d) = 1$ then at least one of primes* 5,7,61,1861 *splits in K.*

Theorem 1.3. *If $\kappa(d) = 2$ and $d \not\equiv 5 \pmod 8$ then at least one of primes* 5,7,61,601,1861 *splits in K.*

By combining theorem 1.1 and theorem 1.2, we can find the upper bound of d with caliber number 1 and from theorem 1.1 and theorem 1.3 we also can find the upper bound of d such that $d \not\equiv 5 \pmod 8$ and $\kappa(d) = 2$. From this, we completely determine all real quadratic fields with caliber number 1 and find all d with $\kappa(d) = 2$ and $d \not\equiv 5 \pmod 8$.

2. Lower bound of caliber number

For each positive integer A, we associate a set

$$S_D(A) := \{[B] \in \mathbb{Z}/2A\mathbb{Z} | B^2 \equiv D \pmod{4A}\}$$

and let $\rho_D(A)$ be the cardinality of $S_D(A)$.

Then we can write a lower bound and an upper bound of $\kappa(d)$ in terms $\rho_D(A)$ as follows:

Proposition 2.1.

$$\sum_{A<\frac{\sqrt{D}}{2}} \rho_D(A) \leq \kappa(D) \leq \sum_{A<\sqrt{D}} \rho_D(A).$$

$\rho_D(A)$ has the following properties.

Proposition 2.2. *1) If $(n,m)=1$, $\rho_D(nm)=\rho_D(n)\rho_D(m)$.*
2) For $p \not| D$,

$$\rho_D(p^\alpha)=1+\chi_D(p),$$

where χ_D be the Kronecker character(ie. $\chi_D(\cdot)=(\frac{D}{\cdot})$). For $p|D$,

$$\rho_D(p^\alpha)=\begin{cases}0, & \alpha>1\\ 1, & \alpha=1\end{cases}$$

Above proposition gives a lower bound of the caliber number of a real quadratic field in terms of D and a rational prime that splits above.

Theorem 2.3. *Let d be a positive square free integer and $K=\mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D. Suppose a rational prime p splits in K.*

$$\kappa(d)>2\Big[\frac{\log\frac{\sqrt{D}}{2}}{\log p}\Big].$$

3. A behavior of Hecke's L-function at $s=0$

Let $d=f(n)$ be a positive square free integer with $f(x)\in\mathbb{Z}[x]$ and $K=\mathbb{Q}(\sqrt{d})$ and q be an integer and $I(q)$ be a group of fractional ideals prime to q and $P_+(q)$ be a group of principal ideals (α) for the totally positive $\alpha\in K$ with $\alpha\equiv 1 \bmod q$. Then we say a multiplicative character $\chi: I(q)/P_+(q)\to\mathbb{C}^*$ is a ray class character modulo q. We define

$$L_K(s,\chi):=\sum_{\substack{\mathfrak{a}\,\in\, I(q)\\ integral}}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

If for $n=qk+r$, $L_K(0,\chi)$ is expressed as

$$L_K(0,\chi)=\frac{1}{24q^2}(A_\chi(r)+B_\chi(r)k)$$

for $A_\chi(r),B_\chi(r)\in\mathbb{Z}[\chi(\mathfrak{a}),\mathfrak{a}\in I(q)/P_+(q)]$,
then we say $L_K(0,\chi)$ is **linear.**

By computing $L(0,\chi)$ explicitly, we find the followings:

Proposition 3.1. *If d is a square free integer of the form $n^2+1,n^2+4,n^2\pm 2$ and $h(d)=1$ then $L(0,\chi)$ is linear.*

Proof. See Lemma1 in [1] and Lemma1 in[2] and Corollary 2.9 and Corollary 3.4 in [4]. ☐

4. Splitting primes in K

Let $\chi : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^*$ be an odd Dirichlet character and ψ is a ray class character defined by $\psi = \chi \circ N : I(q)/P_+(q) \to \mathbb{C}^*$ and $\chi_D = (\frac{D}{\cdot})$ be a Kronecker character. Then we have

$$L_K(0,\psi) = L(0,\chi)L(0,\chi\chi_D)$$
$$= (\frac{1}{q}\sum_{a=1}^{q} a\chi(a))(\frac{1}{qD}\sum_{b=1}^{qD} b\chi(b)\chi_D(b).)$$

Thus if $L_K(0,\psi)$ is linear then

$$B_\psi(r)k + A_\psi(r) = 24q \cdot (\sum_{a=1}^{q} a\chi(a)) \cdot \Big(\frac{1}{qd}\sum_{b=1}^{qd} b\chi(b)\chi_d(b)\Big).$$

Since $\frac{1}{qd}\sum_{b=1}^{qd} b\chi(b)\chi_d(b)$ is an algebraic integer, for a prime ideal I dividing $(\sum_{a=1}^{q} a\chi(a))$, we have

$$B_\psi(r)k + A_\psi(r) \equiv 0 \pmod{I}.$$

And if $I \not| B_\psi(r)$ then

$$k \equiv -\frac{A_\psi(r)}{B_\psi(r)} \pmod{I}$$

Since $n = qk + r$, we have

$$n \equiv -q\frac{A_\psi(r)}{B_\psi(r)} + r \pmod{I}.$$

Moerover if $O_{L_\chi}/I = Z/pZ$ then we can find a unique $T(r) \in \{0, 1, 2, ...p-1\}$ such that

$$-q\frac{A_\psi(r)}{B_\psi(r)} + r + I = T(r) + p\mathbb{Z}.$$

Finally we find the the residue of $n = qk + r$ modulo p.

Now we arrange above all conditions of q and p.

Condition(*)
1. q: odd integer
2. p: odd prime
3. χ: character with conductor q
4. I: prime ideal lying over p
$I \vert (\sum_{a=1}^{q} a\chi(a))$ and $O_{L_\chi}/I = Z/pZ$

For q and p satisfying Condition(*), we denote by

$$q \to p.$$

There are may q and p pairs such that $q \to p$. In section4 of [1] and section 4 of[4], we find

$$175 \to 61,\ 61 \to 1861,\ 175 \to 1861,\ 175 \to 601.$$

Since $L_K(0,\chi)$ is linear for $K = \mathbb{Q}(n^2+2)$ with class number 1, if $q \to p$ then we can find the residue of $n = qk + r$ modulo p shch that $h(n^2+2) = 1$. Assume $h(n^2+2) = 1$ and $n = 175k + 36$. Then we find that

$$n \equiv 28 \pmod{61}.$$

Thus

$$\begin{aligned}\left(\frac{n^2+2}{5}\right) &= \left(\frac{36^2+2}{5}\right) = -1\\ \left(\frac{n^2+2}{7}\right) &= \left(\frac{36^2+2}{7}\right) = -1\\ \left(\frac{n^2+2}{61}\right) &= \left(\frac{28^2+2}{61}\right) = -1\end{aligned}$$

If $n = 175k+36$ with $h(n^2+2) = 1$ then primes 5, 7, 61 inert in $K = \mathbb{Q}(\sqrt{n^2+2})$. For $n = 61k + 28$, we have

$$n \equiv 458 \pmod{1861}.$$

Thus

$$\left(\frac{n^2+2}{1861}\right) = \left(\frac{458^2+2}{1861}\right) = 1.$$

Hence if $n = 175k + 36$ with $h(n^2+2) = 1$ then primes 1861 splits in $K = \mathbb{Q}(\sqrt{n^2+2})$. Finally we find a prime p splits in $K = \mathbb{Q}(\sqrt{n^2+2})$ where $n = 175k + 36$ and $h(n^2+2) = 1$.

In this way, we find that

Proposition 4.1. *If $h(n^2+1) = 1$ or $h(n^2+4) = 1$ then at least one of primes* 5,7,61,1861 *splits in K.*

Proof. See [1],[2]. □

Proposition 4.2. *If $h(n^2 \pm 2) = 1$ then at least one of primes* 5,7,61,601,1861 *splits in K.*

Proof. See section 4 in [4]. □

Moreover by simple computation, we obtain that

Proposition 4.3. *If $\kappa(d) = 1$, then $d = n^2+1$ or n^2+4 and $h(d) = 1$.*

Proof. See section 3 of [5] □

Proposition 4.4. *If $\kappa(d) = 2$ and $d \not\equiv 5 \pmod{8}$, then $d = n^2 \pm 2$ and $h(d) = 1$.*

Proof. See section 3 of [5] □

Finally from Proposition 4.3 and 4.1, we have

Theorem 4.5. *If* $\kappa(d) = 1$ *then at least one of primes* 5,7,61,1861 *splits in* K.

From Proposition 4.4 and 4.2, we also have

Theorem 4.6. *If* $\kappa(d) = 2$ *and* $d \not\equiv 5 \pmod 8$ *then at least one of primes* 5,7,61,601,1861 *splits in* K.

5. Determination of real quadratic fields with small caliber number

Let d be a square free integer and D be the discriminant of $K = \mathbb{Q}(\sqrt{d})$. By combining theorem 2.3 and theorem 4.5, we can obtain the upper bound of d with caliber number 1. If $\kappa(d) = 1$ then $7.5 \times 10^3 > D$. From this, we completely determine all real quadratic fields with caliber number 1.

Theorem 5.1. *(*$\kappa(d) = 1$*) Suppose* d *is a positive square free integer. Then* $\kappa(d) = 1$ *if and only if* d *is one of the following:* $2, 13, 29, 53, 173, 293$.

By combining theorem 2.3 and theorem 4.6, we can obtain the upper bound of d with caliber number 2 where $d \not\equiv 5 \pmod 8$. If $\kappa(d) = 2$ and $d \not\equiv 5 \pmod 8$ then $1.4 \times 10^7 > D$. we list all real quadratic fields of caliber number one and all real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ when $d \not\equiv 5$ modulo 8 with caliber number two.

Theorem 5.2. *(*$\kappa(d) = 2$ *with* $d \not\equiv 5\ (8)$*) Real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with* $d \not\equiv 5$ *modulo* 8 *with caliber number* 2 *are the followings:*

$$3, 6, 11, 38, 83, 227.$$

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