Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 10, Number 1, 2008, pages 85–87

Toric Topology Workshop
KAIST 2008

CLOSED SURFACES WITH LOCALLY STANDARD $(\mathbb{Z}_2)^2$-ACTIONS

BO CHEN

1. Abstract / Introduction

For a Euclidean space $\mathbb{R}^n$, there is a natural $(\mathbb{Z}_2)^n$-action generated by reflections according to the coordinates hyperplanes:

$$(g_1, ..., g_n) : (x_1, ..., x_n) \longmapsto ((-1)^{g_1} x_1, ..., (-1)^{g_n} x_n).$$

A $(\mathbb{Z}_2)^n$-action on an n-manifold M^n is locally standard (ref. [LM]) if locally it looks like the natural transformation on some stable open subset of $\mathbb{R}^n$.

A fact is that for each manifold with locally standard 2-torus action, its orbit space is a nice manifold with corners (ref. [D]). Also, each such action will give a characteristic function (ref. [DJ]) on the set of the facets of the orbit space.

Simple convex polytopes naturally have nice corner structures. In [DJ], Davis and Januszkiewicz studied the locally standard actions with simple polytopes as their orbit spaces, and got many beautiful results.

Here I want to investigate the locally standard actions on connected closed surfaces. The example 1.20 in [DJ] infers that every closed surface admits at least one locally standard action.(For the 2-sphere, there is a natural locally standard action whose orbit space is a 2-gon.)

2000 *Mathematics Subject Classification.* 57M60, 57S17.
Key words and phrases. locally standard action, corner structure, closed surface.

Instead of looking for all the actions on a given closed surfaces, I focus on their orbit spaces.(Notice that different locally standard actions on a certain closed surface may induce different orbit spaces.) Their orbit spaces are nice 2-manifold with corners, described as follows:

Firstly, a surface with boundary can be obtained by deleting some disjoint disks on a connected closed surface, i.e., $Q^2 = S \setminus \bigsqcup_1^k D^2$, where S is a connected closed surface, and $\bigsqcup_1^k D^2$ is a disjoint union of k 2-disks on S. On the other hand, Q^2 can be viewed as the connect sum (at the interior part) of S and k disjoint disks.

Next, one can give a corner structure on Q^2 by making each connected component ($\approx$ circle) of boundary to be a 1-skeleton of a m_i-gon. Color (denote by λ_i) each 1-skeleton of m_i-gon by using nonzero elements in $(\mathbb{Z}_2)^2$. Then we get a nice 2-manifold with corners and an associated characteristic function, denoted by (Q^2, λ), where $\lambda = \{\lambda_i\}$.

The reason that to investigate the locally standard actions can be reduced to studying the orbit space, is that we have the following constructions and a theorem:

Given a nice manifold Q^n with corners and a characteristic function λ (Notice that given a nice manifold with corners, there may not exist any characteristic function), a closed manifold can be constructed (ref. [DJ]) in the following way such that it admits a locally standard $(\mathbb{Z}_2)^n$-action:

$$M(Q, \lambda) := Q \times (\mathbb{Z}_2)^n / \sim,$$

where $(p, g) \sim (q, h)$ iff $p = q$ and $g^{-1}h \in G_{F_p}$.

More generally, given a principe $(\mathbb{Z}_2)^n$-bundle $\xi = (E, \pi, Q)$, a general construction can be done:

$$M(Q, \lambda, \xi) := E / \sim_\xi,$$

where $x \sim_\xi y$ iff $\pi(x) = \pi(y)$ and $\exists g \in G_{F_{\pi(x)}}$, such that $xg = y$.

Remark 1. *When choosing ξ to be a trivial bundle, $M(Q^n, \lambda, \xi) = M(Q^n, \lambda)$.*

The theorem (ref. lemma 3.1 in [LM] or proposition 1.8 in [DJ]) claims that each manifold with a locally standard action can be constructed in this way.

Let's go back to the surface case. Firstly, I want to deal with the trivial principle bundle case: $M(Q^2, \lambda)$. The idea is that cut Q^2 into $k+1$ parts: (0) the surface S with k disks deleted, and (i) m_i-gon D_i^2 with a disk (do not intersect with the boundary) deleted, $i = 1, ..., k$. By the construction, it's easy to see that $M(Q^2, \lambda)$ can be cut to be a disjoint union of 4 copies of part(0) and the small cover (ref.

[DJ]) $M(D_i^2, \lambda_i)$ with 4 disjoint disks deleted. Then,

$$M(Q^2, \lambda) \approx (\sharp S)^4 \sharp M(D_1^2, \lambda_1) \sharp_{i=2}^k (M(D_i^2, \lambda_i) \sharp T^2 \sharp T^2 \sharp T^2),$$

where $(\sharp S)^4$ means the connect sum of 4 copies of the surface S.

Using the result in [NN], the orientation of $M(D_i, \lambda_i)$ can be determined immediately. Then the topological type of $M(Q^2, \lambda)$ can be determined: its Euler characteristic is $4\chi(S) - 4k - \sum_{i=1}^k m_i$ and it's orientable if and only if the surface S is orientable and the image of each λ_i has only two elements (see also [LM]).

We can also give a natural locally standard action on this description.

For general case, [LM] has shown the Euler characteristic of $M(Q^2, \lambda, \xi)$:

$$\chi(M(Q^2, \lambda, \xi)) = 4\chi(Q) - \sum_{i=1}^k m_i = 4\chi(S) - 4k - \sum_{i=1}^k m_i.$$

Notice that the Euler characteristic of the construction is independent of the choice of principle bundle (there may be no nontrivial principle $(\mathbb{Z}_2)^2$-bundle over some Q^2). On the other hand, the choice of principle bundles has nothing to do with the orientation of the construction. So the topological type of the construction is independent of the choice of ξ, i.e.,

$$M(Q^2, \lambda, \xi) \approx M(Q^2, \lambda).$$

But the locally standard action on $M(Q^2, \lambda, \xi)$ depends on ξ. And it's difficult to describe this action, because it's not easy to express a nontrivial principle bundle evidently.

There is a simple application. After analyzing the formula of the Euler characteristic, it's easy to see that there is only one locally standard action on 2-sphere, $\mathbb{R}P^2$, torus and Klein bottle respectively.

References

[DJ] Michael W. Davis and Tadeusz Januszkiewicz, *Convex Polytopes, Coxeter Orbifolds and Torus Actions*, Duke Mahtematicial Journal vol.62 No.2 (1991).

[D] Michael W. Davis, Groups generated by reflections and aspherical mani- folds not covered by Euclidean space, Annals of Mathematics, 117(1983), 293-324.

[LM] Zhi Lü and Mikiya Masuda, *Equivariant classification of 2-torus manifolds*, arXiv:0802.2313

[LY] Zhi Lü and Li Yu, *Topological types of 3-dimensional small covers* , arXiv:0710.4496

[NN] Nakayama, H. and Nishimura, Y., *The orientability of small covers and coloring simple polytopes*, Osaka J. Math. 42(2005),243-256.