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SMALL COVERS OVER CUBE

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A *Small cover* is a closed n-dimensional manifold with a locally standard mod 2 torus $\mathbb{Z}_2^n$ action over a simple convex polytope, which is defined by Davis and Januszkiewicz in [4]. Let P be a simple convex polytope of dimension n and $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ be the set of facets of P. Consider $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ which satisfies the *non-singularity condition*; $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ is a basis of $\mathbb{Z}_2^n$ whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_n}$ is non-empty. We call λ a *characteristic function.* It is well-known that one may assign a characteristic function to a small cover. Two small covers M_1 and M_2 are said to be *weakly* $\mathbb{Z}_2^n$*-equivariantly homeomorphic* (or simply *weakly* $\mathbb{Z}_2^n$*-homeomorphic*) if there is an automorphism $\varphi : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in \mathbb{Z}_2^n$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are $\mathbb{Z}_2^n$-homeomorphic. Following Davis and Januszkiewicz, two small covers M_1 and M_2 over P are said to be *Davis-Januszkiewicz equivalent* (or simply, *D-J equivalent*) if there is a weakly $\mathbb{Z}_2^n$-homeomorphism $f : M_1 \to M_2$ covering the identity on P. By [4], all small covers over P are distinguished their characteristic function λ up to $\mathbb{Z}_2^n$-homeomorphism covering the identity on P, see [4] or [1] for details.

Let $cf(P)$ denote the set of all characteristic functions over P. There are two natural actions on $cf(P)$. One is the free left action of general linear group $GL(n, \mathbb{Z}_2)$ on $cf(P)$ defined by $\sigma \times \lambda \mapsto \sigma \circ \lambda$, where $\lambda \in cf(P)$ and $\sigma \in GL(n, \mathbb{Z}_2)$. An *automorphism* of $\mathcal{F}(P)$ is a bijection from $\mathcal{F}(P)$ to itself which preserves the poset

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structure of all faces of P. Let $\mathrm{Aut}(\mathcal{F}(P))$ denote the group of automorphisms of $\mathcal{F}(P)$. Then there is the right action of $\mathrm{Aut}(\mathcal{F}(P))$ on $cf(P)$ by $\lambda \times h \mapsto \lambda \circ h$, where $\lambda \in cf(P)$ and $h \in \mathrm{Aut}(\mathcal{F}(P))$. Note that there are two one-to-one correspondences

$$\begin{aligned} GL(n, \mathbb{Z}_2) \backslash cf(P) &\longleftrightarrow \{\text{D-J classes over } P\} \\ cf(P)/\mathrm{Aut}(\mathcal{F}(P)) &\longleftrightarrow \{\mathbb{Z}_2^n\text{-homeo. classes over } P\}. \end{aligned}$$

Thus we can count the number of small covers over P by counting the number of orbits of each actions on $cf(P)$.

When P is an n-dimensional cube I^n, one may regard a D-J equivalence of small covers as an $(n \times 2n)$-matrix Λ over $\mathbb{Z}_2$ of form

$$\Lambda = (E_n | \Lambda_*),$$

where E_n is an identity matrix of size n and Λ_* is an $n \times n$ matrix all of whose principal minors are 1, see [3] or [5] for details. Let $M(n)$ be the set of $\mathbb{Z}_2$-matrices of size n all of whose principal minors are 1 and $\mathcal{G}_n$ be the set of acyclic simple digraphs with labeled n nodes. In [3], we have a bijection $\phi : \mathcal{G}_n \to M(n)$ by

$$\phi : G \mapsto A(G) + E_n$$

where $A(G)$ is the vertex adjacency matrix of G and E_n is an identity matrix of size n. Since the cardinality of $\mathcal{G}_n$ is well-known, we have the recursive formula of the number of D-J classes of small covers over cubes. Let R_n be the number of acyclic digraphs with labeled n nodes.

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}.$$

By a quiet similar method, we can establish the formula of the number of D-J classes over a product of simplices in terms of acyclic graphs. Let $\sharp DJ(\prod_{i=1}^{\ell} \Delta^{n_i})$ denote the number of D-J equivalence classes over $\prod_{i=1}^{\ell} \Delta^{n_i}$. Then

$$\sharp DJ(\prod_{i=1}^{\ell} \Delta^{n_i}) = \sum_{G \in \mathcal{G}_\ell} \prod_{v_i \in V(G)} (2^{n_i} - 1)^{\mathrm{outdeg}(v_i)},$$

where $V(G) = \{v_1, \ldots, v_\ell\}$ is the labeled vertex set of G.

On the other hand, recall the criterion for determining the orientability of small covers in [6]; *M is orientable if and only if the sum of entries of i-th column of Λ_* is odd for all $i = 1, \ldots, n$.* Combining the criterion with the above bijection ϕ, we have that M is orientable if and only if $\phi^{-1}(\Lambda_*)$ is the acyclic graph with labeled n nodes all of whose vertices have even indegrees. Let O_n be the number of D-J equivalence classes of orientable small covers over I^n. Then

$$O_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(k-1)(n-k)} R_{n-k}.$$

Note that the ratio O_n/R_n converges to 0 as n increases, see [2] for details.

Finally, we can count the $\mathbb{Z}_2^n$-homeomorphism classes over I^n ([3]). Let Q_n be the number of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over I^n. Then

$$Q_n = \frac{\sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} R_k}{2^n n!} \cdot \prod_{i=0}^{n-1} (2^n - 2^i).$$

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