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COHERENT ELEMENTS AND FIRST LAYERS OF $\mathbb{Z}_p$-EXTENSIONS

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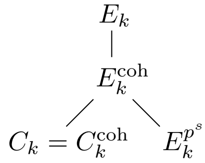
1. Introduction

In [3], some examples of abelian fields k are found such that $E_k^{\text{coh}} \neq C_k$ for some prime p. We briefly recall the construction used in *loc.cit* to obtain an index formula when p is prime to $h_{k^+}[k : \mathbb{Q}]$ where h_{k^+} is the class number of k^+. This case is more simple. We now explain his construction on the condition that p is the only prime of $\mathbb{Q}$ which ramifies in $k/\mathbb{Q}$. Then the inertia field $I(k)$ of k at p must be unramified and abelian over $\mathbb{Q}$. Thus $I(k)$ is equal to $\mathbb{Q}$ and $C_k = C_k^{\text{coh}}$ since $(C_k : C_k^{\text{coh}}) = (C_{I(k)} : C_{I(k)} \cap C_k^{\text{coh}})$ from the proof Lemma 2.5 of [1]. This implies $(E_k : E_k^{\text{coh}}) < \infty$ as well. When $(C_k : C_k^{\text{coh}}) < \infty$ is satisfied, it follows from the remarks after Corollary 3.8 that the coherent units of k is equal to the images of the norm maps of the global units of intermediate fields of degree p^s for all sufficiently large $s \gg 0$. Then, the group E_k^{coh} of coherent units contains $E_k^{p^s}$, for all such s and hence the index $(E_k : E_k^{\text{coh}})$ is a power of p.



On the other hand, the class number formula of Sinnott shows that $(E_k : C_k) = h_{k^+}c_{k^+}$, where c_{k^+} is a constant. However, the constant c_{k^+} vanishes up to a power of 2 when there is only one prime which ramifies in k. Hence, the index $(E_k : C_k)$

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is prime to p and, $E_k = E_k^{\text{coh}}$. Finally, up to a power of 2,

$$(E_k^{\text{coh}} : C_k^{\text{coh}}) = h_{k^+}$$

when p is the only prime of $\mathbb{Q}$ which ramifies in $k/\mathbb{Q}$ and p is prime to $h_{k^+}[k:\mathbb{Q}]$.

We will introduce a guess which relates the coherent property to a characterizing of the first layers of $\mathbb{Z}_p$-extensions of an abelian field. Let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of $k_0 = k = k(\mu_p)$ with k_n its unique subfield of degree p^n over k. We define $\tilde{k}_\infty$ as the inverse limits of $k_n^\times$ with respect to the norm maps. Let π be the natural projection from $\tilde{k}_\infty$ into k. Let k^{coh} denote the group $\pi(\tilde{k}_\infty)$ of coherent elements of k and $k^{\text{univ}} = \cap_n N_n k_n^\times$ be the universal norms in k_∞, where N_n denotes the norm map from k_n to k. For an extension field K/k and a group H, we will say K is H-extendable over k if there is an extension field $F \supset K$ such that F/k is Galois and its Galois group $G(F/k)$ is isomorphic to H. Let Θ_k be the set of all elements α in $k^\times$ such that $k(\sqrt[p]{\alpha})$ is $\mathbb{Z}_p$-extendable. In their paper [2], Bertrandias and Payan studied Θ_k in terms of other subgroups of the ground field. Among these subgroups are the group of universal norms, the group of p-units. We briefly mention the main results of *loc.cit* here. The first main result is a characterizing of elements whose p-th roots generate $\mathbb{Z}/p^n\mathbb{Z}$-extendable fields.

Theorem 1.1 (=Théorème 1 of *loc.cit*)**.** *Let $\alpha \in k^\times$. Then $k(\sqrt[p]{\alpha})$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable if and only if $\alpha \in (k^\times)^p N_n k_n^\times$.*

Based on Theorem 3.11 above, Ψ_k is defined to be the set of all $\alpha \in k^\times$ such that $k(\sqrt[p]{\alpha})$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n. As an immediate corollary of Theorem 3.11 above,

Corollary 1.2 (=Corollary of *loc.cit*)**.**

$$\Psi_k = \bigcap_n (k^\times)^p N_n k_n^\times$$

Using these results, it was shown that an extension $k(\sqrt[p]{\alpha})$ which is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n need not be $\mathbb{Z}_p$-extendable by showing an example of k such that $\Theta_k \neq \Psi_k$. We are now ready to introduce our guess.

Guess. *For an abelian field k, $\Theta_k = k^{\text{coh}} k^{\times p}$.*

At the present moment, we do not know whether the above equality holds for all abelian fields. One inclusion follows from the same arguments of the proof of

Proposition 1.3 of *loc.cit.* Hence, for an abelian field k,

$$\Theta_k \supset k^{\mathrm{coh}} k^{\times p}.$$

We will find a condition on the prime p under which the above guess is true. More precisely, if p is the only prime which ramifies in k such that the Sylow p-subgroup of the class group of k is generated by the class of $\mathfrak{p}$, then $\Theta_k = k^{\mathrm{coh}} k^{\times p}$. We need the following result of *loc.cit.*

Proposition 1.3 (=Théorème 3 of *loc.cit*)**.** *For a number field $k = k(\mu_p)$, if there is only one prime $\mathfrak{p}$ in k lying over p and the Sylow p-subgroup of the class group of k is generated by the class of $\mathfrak{p}$, then*

$$\Theta_k = \Psi_k = E_k^{(\mathrm{p})} k^{\times p}.$$

Notice that from the assumption on p, there is a norm coherent sequence $(\pi_n)_{n\in\mathbb{N}}$ of prime elements π_n of k_n lying over p. It follows from the previous remarks that

$$k^{\mathrm{coh}} = (E_k^{(p)})^{\mathrm{coh}} = (E_k^{(p)})^{\mathrm{univ}}.$$

By the Sinnott's class number formula, if the assumption of Proposition 3.13 is satisfied, then $C^{(\mathrm{p})} k^{\times p} \otimes \mathbb{Z}_p = E_k^{(\mathrm{p})} k^{\times p} \otimes \mathbb{Z}_p$ and hence

$$C^{(\mathrm{p})} k^{\times p} \otimes \mathbb{Z}_p = (E_k^{(p)})^{\mathrm{coh}} k^{\times p} \otimes \mathbb{Z}_p$$

since $(C_k^{(p)})^{\mathrm{coh}} = (C_k^{(p)})$. From the equality $(E_k^{(p)})^{\mathrm{coh}} k^{\times p} = k^{\mathrm{coh}} k^{\times p}$, it follows that

$$\Theta_k = E_k^{(\mathrm{p})} k^{\times p} = C_k^{(\mathrm{p})} k^{\times p} = k^{\mathrm{coh}} k^{\times p}.$$

This proves that if there is only one prime $\mathfrak{p}$ in k lying over p such that the Sylow p-subgroup of the class group of k is generated by the class of $\mathfrak{p}$, then our guess is true.

References

[1] J. Belliard, Sous-modules d'unites en theorie d'Iwasawa, Theorie des nombres, Annees 1998/2001, Publ. Math. UFR Sci. Tech. Besancon, Univ. Franche-Comte, Besancon(2002), 12 pp.

[2] F. Bertrandias and J. Payan, Γ-extensions et invariants cyclotomiques, Ann. Sci. Ecole Norm. Sup. 5(1972), 517-543.

[3] R. Coleman, On an Archimedean characterization of the circular units, J. reine angew. Math. 356(1985), 161-173.

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