REMARKS
ON MCGAVRAN’S PAPER AND NISHIMURA’S RESULT

SHINTARÔ KUROKI

This is the extended abstract for the informal seminar during the Buchstaber’s lecture in KAIST, and summarizes the paper [K].

In the seminar, we focus on oriented small covers (i.e., its first Stiefel-Whitney class is zero; \( w_1(M^n) = 0 \)) and spin quasitoric manifolds (i.e., \( w_2(M^{2n}) = 0 \)).

Let \( \Lambda(k) = (I_n \mid \Lambda'(k)) \) be a characteristic function on an \( n \)-dimensional simple polytope \( P \), where \( k = \mathbb{Z}_2 \) or \( \mathbb{Z} \) (see [DJ91] or [BP02] for detail). By using the method of [NN05], we can easily show the following proposition for these manifolds.

**Proposition 0.1.** If \( M(P, \Lambda(k)) \) is an oriented \( n \)-dimensional small cover for \( k = \mathbb{Z}_2 \) or a spin \( 2n \)-dimensional quasitoric manifold for \( k = \mathbb{Z} \), then each row vector \( (\lambda_1, \lambda_2, \cdots, \lambda_n)^t \in k^n \) in \( \Lambda'(k) \) satisfies that

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_n \equiv 1 \pmod{2}.
\]

Therefore, if \( r \) is odd, the following 3-dimensional quasitoric manifold (Figure 1) has a spin structure (also see examples in [CMS]).

McGavran in [M76] studied 6-dimensional spin manifolds with \( T^3 \)-actions. According his lemma (4.9 Lemma of [M76]), the above \( r \) in Figure 1 must be 0 or 1. However, if \( |r| > 2 \), the above example does not satisfy the McGavran’s lemma.

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It follows that there is a mistake in McGavran’s lemma (4.9 Lemma in [M76]).
However, we can use his argument for small covers.

3-dimensional small covers have been studied by Izmestiev in [I01] for 3-colorable cases, Nakayama and Nishimura in [NN05] and [N04] for oriented cases (i.e., 4-colorable cases), and Lü and Yu in [LY] for all cases. In particular, Nishimura in [N04] shows the following theorem.

**Theorem 0.2** (Nishimura). *Let M be an oriented 3-dimensional small cover. Then M can be constructed from the real projective space \( \mathbb{R}P(3) \) and the 3-dimensional torus \( T^3 \) by using finite times equivariant connected sums \( \# \) and 2 equivariant surgeries \( \sharp \) and \( \flat \).

Here, \( \sharp \), \( \sharp \) and \( \flat \) are as in Figure 2.

**Figure 2.** The first is the equivariant connected sum \( \sharp \), the second is the equivariant surgery \( \flat \), and the third is the another equivariant surgery \( \flat \).

Using the McGavran’s idea, we can improve Nishimura’s Theorem 0.2 as the following main theorem.

**Theorem 0.3.** *Let M be an oriented 3-dimensional small cover. Then M can be constructed from the real projective space \( \mathbb{R}P(3) \) and the 3-dimensional torus \( T^3 \) by using finite times equivariant connected sums \( \sharp \) and equivariant surgeries \( \sharp \).

In other words, the equivariant surgery \( \flat \) can be constructed by \( \sharp \) and \( \sharp \) as Figure 3.
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Figure 3. $\mathfrak{b} = (\mathfrak{z}\Delta)^{-1} \circ \mathfrak{z}^{-1}$ and $\mathfrak{b}^{-1} = \mathfrak{z} \circ (\mathfrak{z}\Delta)$.

REFERENCES


School of Mathematical Science Fudan University, Shanghai, 200433, P.R. China
E-mail address: kuroki@fudan.edu.cn