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MOTIVIZATION FOR ADDITIVE K-THEORY

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This article is based on a talk given by the author at KAIST Algebraic Structure and its Applications Research Center (ASARC) Workshop from June 4th - 6th, 2009 at Muju, Korea. The talk aimed to introduce the motivization problem for additive K-theory to wider mathematical audiences, and report some progresses on the problem.

The problem, *motivization for additive K-theory*, is related to a few disciplines of modern algebra, topology, and algebraic geometry.

The starting point of the problem is the following kind of Grothendieck-Riemann-Roch theorem for a smooth variety X over a field k, considered by S. Bloch

$$K_n(X)_{\mathbb{Q}} \simeq \bigoplus_q CH^q(X,n)_{\mathbb{Q}} \tag{0.1}$$

where K_n is the n-th higher algebraic K-group of D. Quillen ([12]), and $CH^q(X,n)$ is the n-th higher Chow group of codimension q-cycles on X of S. Bloch ([2]). After tensoring with $\mathbb{Q}$ over $\mathbb{Z}$, the Quillen K-groups on the left hand side were called the *rational motivic cohomology* by A. Beilinson in [1] as an attempt to get an approximate answer to a grand conjectural program of A. Grothendieck seeking algebro-geometric objects called *motives* that Grothendieck envisioned as objects made of algebraic cycles whose category is abelian containing enough cohomological information of algebraic varieties. Grothendieck believed that these motives should allow us to know answers of various arithmetic questions of algebraic varieties. Unfortunately this search of motives turned out to be highly difficult, and as a detour P. Deligne suggested that one should probably try to develop a theory of triangulated category of motives that may be the derived category of a suitable

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abelian category. After many years V. Voevodsky, together with E. Friedlander and A. Suslin, constructed one such category Deligne predicted, called the triangulated category of geometric motives, and computed suitable Ext groups that form the *integral motivic cohomology*. One of his theorems ([15]) shows that Bloch's higher Chow groups indeed serve the roles of the motivic cohomology groups after suitable reindexing. Thus, the equation (0.1) can be interpreted as a "motivic" description of the Quillen K-group.

Additive K-theory considers an infinitesimal situation of the above in the following sense: when R is an associative algebra, and $H_*(GL(R), \mathbb{Z})$ is its graded Hopf algebra of the discrete group homology groups, Quillen's plus construction and the Hurewicz map then induce the isomorphism after $\otimes\mathbb{Q}$

$$K_n(R)_{\mathbb{Q}} = \mathrm{Prim}_n H_*(GL(R), \mathbb{Z})_{\mathbb{Q}} \tag{0.2}$$

of the n-th Quillen K-group to the n-th graded piece of the primitive part of the Hopf algebra. Additive K-theory is obtained by replacing the group $GL(R)$ by its Lie algebra $\mathfrak{gl}(R)$, and the group homology by the Lie algebra homology $H_*(\mathfrak{gl}(R), \mathbb{Q})$ for a $\mathbb{Q}$-algebra R:

$$K_n^+(R) = \mathrm{Prim}_n H_*(\mathfrak{gl}(R), \mathbb{Q}). \tag{0.3}$$

An interesting point here is that this group turns out to be isomorphic to the $(n-1)$-th cyclic homology $HC_{n-1}(R)$ of A. Connes as proven in [7, 14]. In other words, the cyclic homology is indeed the additive K-theory up to shift of an index.

Our problem in the paper thus means that we want a "motivic" description of the additive K-group when R is a sufficiently good ring over the field of rational numbers. Since we don't yet have a suitable Voevodsky type category that fits into this picture, for a motivic description we hope to construct suitable groups made from algebraic cycles.

What kind of groups of algebraic cycle origin should we take? This is not yet an easy question, but we are able to make a list, likely be incomplete, of properties that show what we should aim to achieve.

The first attempt of the author was to use the *additive* higher Chow groups defined first by S. Bloch and H. Esnault in [3, 4]. One undesirable feature turns out to be that, while the group of 0-cycles for higher Chow groups describes the Milnor K-groups

$$K_n^M(k) \simeq CH_0(k, n) \tag{0.4}$$

as seen in [9, 13], the corresponding group of 0-cycles for additive higher Chow groups describes the absolute Kähler differentials

$$\Omega^n_{k/\mathbb{Q}} \simeq ACH_0(k,n) \tag{0.5}$$

as seen in [4]; known calculations of the cyclic homology groups clearly show that the additive K-groups behave more like part of de Rham cohomology groups, not the groups of differentials themselves. This imcompatibility seems to destroy the plan of our journey from the starting point, however this is actually due to an inappropriate usage of the terminology *additive* in the additive higher Chow groups.

The correction comes from the basic fact that, the cyclic homology groups always accompany the Hochschild homology groups, and vice versa. The description of the Hochschild homology groups as the primitive parts of some graded Hopf algebra was studied by C. Cuvier and J.-L. Loday in [5, 8]. For expressions such as (0.2) and (0.3) for the Hochschild homology, one actually has to consider the *Leibniz algebra*, a noncommutative version of Lie algebras, and the *Leibniz homology*, a noncommutative version of Lie algebra homology groups. These Leibniz homology groups of the Lie algebra $\mathfrak{gl}(R)$, considered as a Leibniz algebra, give a graded noncommutative cocommutative Hopf algebra, whose graded primitive part gives the isomorphism

$$HH_{n-1}(R) = \mathrm{Prim}_n HL_*(\mathfrak{gl}(R),\mathbb{Q}). \tag{0.6}$$

This suggests that one may regard the Hochschild homology as a noncommutative additive K-theory, while the cyclic homology as a commutative additive K-theory. Cuvier and Loday proved a parallel list of properties for Leibniz homology and Hochschild homology groups.

A motivization problem for additive K-theory should, consequently, look for two different kind of complexes of algebraic cycles, one for the Hochschild homology the other for the cyclic homology. It turns out that Bloch-Esnault's additive higher Chow groups, contrary to their name *additive*, are close to the Hochschild side, while it is possible to deduce a cyclic analogue of higher Chow groups that are close to the cyclic side. The following table shows a summary of known results on K-theories and some of new results:
† For char$(k) = 0$, we have $\Omega^{n-1}_{k/\mathbb{Z}} \simeq \Omega^{n-1}_{k/\mathbb{Q}}$

It is a foundational formalism in the theory of cyclic homology that the Hochschild complex induces a bicomplex called the *mixed complex* whose total complex homology is by definition the cyclic homology. If our assertion that the additive higher

Primitive parts ([5, 8, 7, 14])	$K:\quad \mathrm{Prim}_n H_*(GL(R),\mathbb{Z})_{\mathbb{Q}} = K_n(R)_{\mathbb{Q}}$ $HH:\quad \mathrm{Prim}_n HL_*(\mathfrak{gl}(R),\mathbb{Q}) = HH_{n-1}(R)$ $HC:\quad \mathrm{Prim}_n H_*(\mathfrak{gl}(R),\mathbb{Q}) = HC_{n-1}(R)$
Suslin-type stability ([5, 8, 9, **?**])	$K:\quad$ For $m \geq n \quad H_n(GL_m(R),\mathbb{Z}) \xrightarrow{\simeq} H_n(GL(R),\mathbb{Z})$ $HH:\quad$ For $m \geq n \quad HL_n(\mathfrak{gl}_m(R),\mathbb{Q}) \xrightarrow{\simeq} HL_n(\mathfrak{gl}(R),\mathbb{Q})$ $HC:\quad$ For $m \geq n \quad H_n(\mathfrak{gl}_m(R),\mathbb{Q}) \xrightarrow{\simeq} H_n(\mathfrak{gl}(R),\mathbb{Q})$
1st obstruction to stability† ([5, 8, 9, **?**])	$K:\quad H_n(GL_{n-1}(k),\mathbb{Z}) \to H_n(GL_n(k),\mathbb{Z}) \to K_n^M(k) \to 0$ $HH:\quad HL_n(\mathfrak{gl}_{n-1}(k),\mathbb{Q}) \to HL_n(\mathfrak{gl}_n(k),\mathbb{Q}) \to \Omega^{n-1}_{k/\mathbb{Q}} \to 0$ $HC:\quad H_n(\mathfrak{gl}_{n-1}(k),\mathbb{Q}) \to H_n(\mathfrak{gl}_n(k),\mathbb{Q}) \to \Omega^{n-1}_{k/\mathbb{Q}}/d\Omega^{n-2}_{k/\mathbb{Q}} \to 0$
Motivization of 1st obstruction† ([4, 13],Cor.0.4)	$K:\quad K_n^M(k) \simeq CH_0(k,n)$ $HH:\quad \Omega^{n-1}_{k/\mathbb{Q}} \simeq ACH_0(k,n-1)$ $HC:\quad \Omega^{n-1}_{k/\mathbb{Q}}/d\Omega^{n-2}_{k/\mathbb{Q}} \simeq CCH_0(k,n-1)$

Chow complex is really supposed to behave like the Hochschild complex is sound, the first thing one should be able to do is to imitate this mixed complex. The first main result is the construction of a morphism δ, an analogue of the Connes boundary operator B in cyclic homology theory, that gives a double complex which is a mixed complex:

Theorem 0.1. *Let X be a k-variety. Then, the additive higher Chow complex $\{\mathcal{Z}_p(X\times\Diamond_n),\partial\}_{p,n\geq 0}$ has a quasi-isomorphic subcomplex $\{\mathcal{Z}_p(X\times\Diamond_n)_0,\partial'\}_{p,n\geq 0}$ with a motivic analogue of Connes B operators $\delta:\mathcal{Z}_p(X\times\Diamond_n)_0\to\mathcal{Z}_p(X\times\Diamond_{n+1})_0$ (see §**??**) such that the bicomplex $\mathcal{BZ} := \{\mathcal{Z}(n) := \bigoplus_p \mathcal{Z}_p(X\times\Diamond_n)_0,\partial',\delta\}_{n\geq 0}$ forms a mixed complex (see* [8]*):*

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial'} \\
\mathcal{Z}(3) & \xleftarrow{\delta} & \mathcal{Z}(2) & \xleftarrow{\delta} & \mathcal{Z}(1) & \xleftarrow{\delta} & \mathcal{Z}(0) \\
\downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial'} & & \\
\mathcal{Z}(2) & \xleftarrow{\delta} & \mathcal{Z}(1) & \xleftarrow{\delta} & \mathcal{Z}(0) & & \\
\downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial'} & & & & \\
\mathcal{Z}(1) & \xleftarrow{\delta} & \mathcal{Z}(0) & & & & \\
\downarrow{\scriptstyle\partial'} & & & & & & \\
\mathcal{Z}(0) & & & & & &
\end{array}$$

Several interesting immediate corollaries are the following:

Corollary 0.2. The Connes map δ induces a differential δ_* on additive Chow groups: $\delta_* : ACH_p(X,n) \to ACH_p(X,n+1)$. In particular, when $X = \mathrm{Spec}(k)$ and $p = 0$, it is identical to $\delta_* = (n+1)\cdot d : \Omega^n_{k/\mathbb{Z}} \to \Omega^{n+1}_{k/\mathbb{Z}}$.

Letting $CCH_p(X,n)$ be the homology of the total complex of $\mathcal{BZ}$, we have

Corollary 0.3. The groups ACH and CCH form the following long exact sequence

$$\cdots \xrightarrow{B} ACH_p(n) \xrightarrow{I} CCH_p(n) \xrightarrow{S} CCH_{p-1}(n-2) \xrightarrow{B} ACH_{p-1}(n-1) \xrightarrow{I} \cdots,$$

where $ACH_p(n) := ACH_p(X,n)$ and $CCH_p(n) := CCH_p(X,n)$. The maps I, S, B have bidegrees $(0,0), (-1,-2), (0,+1)$ in (p,n), respectively.

The is a kind of the Connes periodicity exact sequence for the Hochschild and cyclic homologies for these groups of algebraic cycles.

Corollary 0.4. $CCH_0(k, n-1) \simeq \Omega^{n-1}_{k/\mathbb{Z}}/d\Omega^{n-2}_{k/\mathbb{Z}}$.

It is also a basic fact that the Connes boundary map B induces a differential structure on the Hochschild homology groups. We have a similar situation for our Connes boundary δ. Limited to the zero-cycles, we can define a graded commutative product structure on algebraic cycles that is compatible with the wedge product of the absolute differential forms. For this wedge, the map δ induces a differential that is a derivation. Since δ is given by a morphism of varieties, it comes from the graph correspondence, which is an algebraic cycle. This answers a question of Bloch and Esnault on describing the exterior derivation of the graded algebra $\Omega^*_{k/\mathbb{Z}}$. A detailed discussion is given in [11]. A stronger result with deeper arguments is given in [6].

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