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ORIENTABILITY OF FACIAL SUBMANIFOLDS OF SMALL COVERS

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Suppose Q^n is a simple convex polytope of dimension n, M^n is a small cover over Q^n, i.e. M^n has a locally standard $(\mathbb{Z}_2)^n$-action whose orbit space is Q^n (see [1]). It is shown in [2] that M is orientable if and only if the image of the characteristic function can be turned into the set $\mathcal{O}^n = \{e_{i_1} + \cdots + e_{i_k} \,|\, e_{i_j} \neq e_{i_{j'}}, k \text{ is odd}\}$ by a basis change of $(\mathbb{Z}_2)^n$.

Let F^{k} be a k-face of Q^n, the closed manifold $p^{-1}(\mathrm{F}^{\mathrm{k}})$ is called the *facial submanifold* of M with respect to the F^{k}. In this talk, we show how to use characteristic function of the small cover to judge the orientability of any facial submanifold in a small cover.

Let F be an $(n-1)$-facet of Q^n and let $\widetilde{\mathrm{F}}_1, \cdots, \widetilde{\mathrm{F}}_{\mathrm{s}}$ be all $(n-1)$-faces adjacent to F in Q^n. Then F is an $(n-1)$-dimensional simple convex polytope whose $(n-2)$-faces are $\mathrm{F} \cap \widetilde{\mathrm{F}}_{\mathrm{j}}$, $j = 1, \cdots s$.

Suppose λ is the characteristic function of the small cover. Let

$$\Phi_{\mathrm{F}} : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n / \langle \lambda(\mathrm{F}) \rangle \cong (\mathbb{Z}_2)^{n-1}$$

be the quotient map. This induced a function on all the $(n-2)$-faces $\mathrm{F} \cap \widetilde{\mathrm{F}}_{\mathrm{j}}$ of $\partial \mathrm{F}$ by: $\lambda(\mathrm{F} \cap \widetilde{\mathrm{F}}_{\mathrm{j}}) := \Phi_{\mathrm{F}}(\lambda(\widetilde{\mathrm{F}}_{\mathrm{j}})) \in (\mathbb{Z}_2)^{n-1}$. We can prove the following statement.

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Proposition 1: $\pi^{-1}(\mathrm{F})$ is an orientable submanifold of M^n if and only if $\lambda(\mathrm{F} \cap \widetilde{\mathrm{F}}_{\mathrm{j}})$ can be turned into the set $\mathcal{O}^{n-1}$ by a basis change of $(\mathbb{Z}_2)^{n-1}$.

Furthermore, for any k-face F^{k} of Q^n, we can choose a descending face sequence on Q^n

$$Q^n \supset \mathrm{F}^{\mathrm{n-1}} \cdots \supset \mathrm{F}^{\mathrm{k+1}} \supset \mathrm{F}^{\mathrm{k}} \tag{1}$$

where F^{j} is a j-face of Q^n. We can inductively define λ on the facets of the boundary of F^{j}. At the end, we can judge the orientability of the facial submanifold $\pi^{-1}(\mathrm{F}^{\mathrm{k}})$ using proposition 1.

As an application, we enumerate 3-dimensional small covers over prisms $Q = P(n) \times I$ where $P(n)$ is a polygon with n-edges. In particular, for those orientable ones, we can show the following:

Proposition 2: Suppose M_1, M_2 are orientable small covers over prisms. If $H^*(M_1) \cong H^*(M_2)$ as graded rings, M_1 must be homeomorphic to M_2.

It is very likely that similar statement holds for non-orientable small covers over prisms too. But so far, the proof is not completed. In general, we have the following problem (also see [3]):

Problem: If M_1 and M_2 are small covers over the same simple convex polytope Q and $H^*(M_1) \cong H^*(M_2)$ as graded rings, is M_1 homeomorphic to M_2?

This problem is called **cohomological rigidity problem** in [4] and has been confirmed to be true for all real Bott tower manifolds there.

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