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HILBERT FUNCTIONS OF STANDARD GRADED ALGEBRAS OVER A FIELD

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Abstract. In this talk, we introduce Hilbert functions of a graded algebras over a field, and one of the long standing conjectures concerning them, Fröberg conjecture. Then we study relations between Fröberg conjecture on Hilbert series and Moreno-Socias Conjecture. Consequently, we show that Fröberg conjecture holds for special cases as an example.

1. Almost Reverse Lexicographic Monomial Ideal

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k. Throughout this talk, we assume that k is a field of characteristic 0. If I is an ideal in R, by a definition, a Hilbert function of R/I is a numerical function from $\mathbb{Z}_{\geq 0}$ into $\mathbb{Z}_{\geq 0}$ defined to be

$$H(R/I, d) = \dim_k R_d - \dim_k I_d,$$

for each d. We use only the reverse lexicographic order as a multiplicative term order. A monomial ideal I in R is said to be *almost reverse lexicographic* if I contains every monomial M which is bigger than a minimal generator of I having the same degree with M. One of conjectures associated with the almost reverse lexicographic ideal is Moreno-Socias conjecture [13].

Conjecture 1.1 (Moreno-Socias)**.** *If I is a homogeneous ideal generated by generic forms in R, then the generic initial ideal* $\operatorname{gin}(I)$ *of I is almost reverse lexicographic.*

Another longstanding conjecture on generic algebras is Fröberg conjecture [7].

Conjecture 1.2 (Fröberg)**.** *If I is a homogeneous ideal generated by generic forms $F_1, \ldots, F_r$ in R of degrees $\deg F_i = d_i$, then the Hilbert series $S_{R/I}(z)$ of R/I is*

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given by

$$S_{R/I}(z) = \left| \frac{\prod_{i=1}^{r}(1-z^{d_i})}{(1-z)^n} \right|.$$

Let I be a homogeneous Artinian ideal in R which has the strong Lefschetz property. For a degree d form $F \in R$ we have the following exact sequence

$$0 \to ((I:F)/I)(-d) \to R/I(-d) \xrightarrow{\times F} (R/I) \to (R/I+(F)) \to 0.$$

If F is generic, then the Hilbert function of $R/I+(F)$ is given by

$$H(R/I+(F),t) = \max\{H(R/I,t) - H(R/I,t-d), 0\}.$$

Hence the Hilbert series of $R/I+(F)$ is given by

$$S_{R/I+(F)}(z) = |(1-z^d)S_{R/I}(z)|.$$

Hence the study for the strong Lefschetz property of standard graded k-algebras defined by generic forms is closely related with the study for Fröberg conjecture. To introduce the relation between Fröberg conjecture and Moreno-Socias conjecture, we need some notations.

Suppose that I is a homogeneous Artinian ideal of $R = k[x_1,\ldots,x_n]$. In the paper [4], Cho et al. showed that the minimal system of generators of $\mathrm{gin}(I)$ is completely determined by the positive integer f_1 and functions $f_i : \mathbb{Z}_{\geq 0}^{i-1} \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined as follows:

$$f_1 = \min\{t \mid x_1^t \in \mathrm{gin}(I)\}, \text{ and} \tag{1.1}$$

$$f_i(\alpha_1,\ldots,\alpha_{i-1}) = \min\{t \mid x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^t \in \mathrm{gin}(I)\}, \tag{1.2}$$

for each $2 \leq i \leq n$.

Proposition 1.3. [4] *Let I be a homogeneous Artinian ideal in $R = k[x_1,\ldots,x_n]$. Suppose that $f_1,\ldots,f_n$ are defined for* $\mathrm{gin}(I)$ *as in* (1.1). *Then the minimal system of generators $\mathcal{G}(\mathrm{gin}(I))$ of* $\mathrm{gin}(I)$ *is*

$$\mathcal{G}(\mathrm{gin}(I)) = \{x_1^{f_1}\} \cup \left\{ x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}x_i^{f_i(\alpha_1,\ldots,\alpha_{i-1})} \;\middle|\; \begin{array}{l} 2 \leq i \leq n, \\ 0 \leq \alpha_1 < f_1, \text{ and} \\ 0 \leq \alpha_j < f_j(\alpha_1,\ldots,\alpha_{j-1}) \\ \text{for each } 2 \leq j \leq i \end{array} \right\}.$$

□

For each $1 \le i \le n-1$, let the set J_i be defined as

$$J_i = \left\{ (\alpha_1, \dots, \alpha_i) \;\middle|\; \begin{array}{l} 0 \le \alpha_1 < f_1, \text{ and} \\ 0 \le \alpha_j < f_j(\alpha_1, \dots, \alpha_{j-1}) \\ \text{for each } 2 \le j \le i \end{array} \right\}. \tag{1.3}$$

For $\alpha = (\alpha_1, \dots, \alpha_i) \in \mathbb{Z}^i_{\ge 0}$, we denote $\sum_{j=0}^{i} \alpha_j$ by $|\alpha|$. And for $\beta = (\beta_1, \dots, \beta_i) \in \mathbb{Z}^i_{\ge 0}$, we say that $\beta > \alpha$ if $x_1^{\beta_1} \cdots x_i^{\beta_i} > x_1^{\alpha_1} \cdots x_i^{\alpha_i}$.

In the paper [2], Ahn et al. gave the following tool detecting whether I has the strong Lefschetz (or Stanley) property, from the view point of the minimal system of generators of $\text{gin}(I)$.

Proposition 1.4. [2] *Let I be a homogeneous Artinian ideal in $R = k[x_1, \dots, x_n]$ with $t = \max\{i \mid (R/I)_i \ne 0\}$.*

(1) *I has the strong Lefschetz property if and only if we have*

$$f_n(0, \dots, 0, |\alpha| + 1) + 1 \le f_n(\alpha),$$

for any $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in J_{n-1}$.

(2) *I has the strong Stanley property if and only if we have*

$$f_n(\alpha) = t - 2|\alpha| + 1,$$

for any $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in J_{n-1}$.

The following proposition, showed by Cho et al. [3], gives an equivalent condition for $\text{gin}(I)$ to be almost reverse lexicographic.

Proposition 1.5. [3] *Let I be a homogeneous Artinian ideal in the polynomial ring $R = k[x_1, \dots, x_n]$. Then $\text{gin}(I)$ is almost reverse lexicographic if and only if for each $2 \le i \le n-1$ the following conditions are satisfied:*

(1) *There exist generic linear forms $L_1, \dots, L_{n-i-1}$ in R such that the ring $R/I + (L_1, \dots, L_{n-i-1})$ has the strong Lefschetz property.*

(2) *For any two elements $\alpha = (\alpha_1, \dots, \alpha_i)$ and $\beta = (\beta_1, \dots, \beta_i)$ of J_i with $|\alpha| = |\beta|$ and $\alpha < \beta$, we have $f_{i+1}(\beta) \le f_{i+1}(\alpha)$.*

From the proposition, we can easily get

Corollary 1.6. *For every homogeneous Artinian ideal K in the polynomial ring $S = k[x_1, x_2]$, $\text{gin}(K)$ is almost reverse lexicographic.*

The following proposition shows that for a homogeneous Artinian ideal in $S = k[x_1, x_2, x_3]$, the second condition in Proposition 1.5 is superfluous, that is, $\text{gin}(I)$ is almost reverse lexicographic if and only if S/I has the strong Lefschetz property.

Proposition 1.7. [3] *Let I be a homogeneous Artinian ideal of $S = k[x_1, x_2, x_3]$. Then S/I has the strong Lefschetz property if and only if* $\operatorname{gin}(I)$ *is almost reverse lexicographic.*

Using this proposition, Cho et al. showed

Theorem 1.8. [3] *Let I be a homogeneous Artinian ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$. Then* $\operatorname{gin}(I)$ *is almost reverse lexicographic if and only if the following conditions are satisfied:*

(1) *For any $0 \leq i \leq n-3$, there exist generic linear forms $L_1, \ldots, L_i$ in R such that the ring $R/I + (L_1, \ldots, L_i)$ has the strong Lefschetz property.*
(2) *For each $3 \leq i \leq n-1$, if $\alpha = (\alpha_1, \ldots, \alpha_i)$, $\beta = (\beta_1, \ldots, \beta_i)$ are two elements of J_i with $|\alpha| = |\beta|$ and $\alpha < \beta$, then we have $f_{i+1}(\beta) \leq f_{i+1}(\alpha)$.*

Hence, we can study Fröberg conjecture by studying the generic initial ideals of the ideals generated by generic forms. From the following theorem, we can show that Fröberg conjecture holds for the case that the ideal generated by generic forms, all of which have degree only 3, in the polynomial ring $S = k[x, y, z]$.

Theorem 1.9. [3] *Let I be a homogeneous ideal in $R = k[x_1, \ldots, x_n]$ which has a almost reverse lexicographic generic initial ideal. If K is a homogeneous ideal generated by generic forms in R such that $H(R/I, d) = H(R/K, d)$ for all d, then* $\operatorname{gin}(K)$ *is also almost reverse lexicographic.*

Example 1.10. Let $S = k[x, y, z]$, and let I_i be the ideal generated by i generic forms of degree 3, for each $i \geq 3$. For I_3, $\operatorname{gin}(I_3) = (x^3, x^2y, xy^2, y^4, y^3z, x^2z^3, xyz^3, y^2z^3, xz^5, yz^5, z^7)$ and its Hilbert function is $H(R/I_3, \bullet) = (1, 3, 6, 7, 6, 3, 1, 0)$. So for I_4, we have $H(R/I_4, \bullet) = (1, 3, 6, 6, 3, 0)$. Since the monomial ideal $(x^3, x^2y, xy^2, y^3, x^2z^2, xyz^2, y^2z^2, xz^4, yz^4, z^5)$ is an ARL monomial ideal having the same Hilbert function as I_4, Theorem 1.9 implies that $\operatorname{gin}(I_4) = (x^3, x^2y, xy^2, y^3, x^2z^2, xyz^2, y^2z^2, xz^4, yz^4, z^5)$.

This implies that $H(R/I_5, \bullet) = (1, 3, 6, 5, 0)$. Note that the monomial ideal $(x^3, x^2y, xy^2, y^3, x^2z, xyz^2, y^2z^2, xz^3, yz^3, z^4)$ is an ARL monomial ideal having the same Hilbert function as I_5. So we have $\operatorname{gin}(I_5) = (x^3, x^2y, xy^2, y^3, x^2z, xyz^2, y^2z^2, xz^3, yz^3, z^4)$ from Theorem 1.9. Continuing this process, we can see that Fröberg conjecture holds for this case.

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