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DIFFEOMORPHISMS WITH LIMIT WEAK SHADOWING

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ABSTRACT. In this paper, we introduce the notion of the limit weak shadowing property which is a really stronger property than the usual weak shadowing property, and show that there is a diffeomorphism f on 2 dimensional torus belonging to the C^1-interior of the set of diffeomorphisms possessing the limit weak shadowing property such that f does not satisfy the strong transversality condition. For the usual weak shadowing property, the existence of such the map has already known.

The weak shadowing property is investigated in [2, 3, 8, and 9], and a remarkable example having the property is treated in [4] at first. Every homeomorphism having the shadowing property has the weak shadowing property but its converse is not true. Indeed, an irrational rotation map ρ on the unit circle has the weak shadowing property but ρ does not have the shadowing property.

The dynamics of diffeomorphisms belonging to the C^1-interior of the set of diffeomorphisms on a C^∞ closed surface having the weak shadowing property is investigated in [3, 8, and 9], and it is proved that such the diffeomorphisms satisfy both Axiom A and no-cycle condition but, in general, they do not satisfy the strong transversality condition.

In this paper, we introduce the notion of the limit weak shadowing property which is a really stronger property than the usual weak shadowing property, and show that there is a diffeomorphism f on 2 torus belonging to the C^1-interior of the set of diffeomorphisms possessing the limit weak shadowing property such that f does not satisfy the strong transversality condition. For the usual weak shadowing property, the existence of such the map has already known.

Let M be a C^∞ closed manifold, and let $\mathrm{Diff}(M)$ be the set of C^1 diffeomorphisms on M endowed with the C^1-topology. Denote by $\mathcal{L}WS(M)$ the set of all $f \in \mathrm{Diff}(M)$ having the limit weak shadowing property, and by $\mathrm{int}\mathcal{L}WS(M)$ its C^1-interior. Then the following is obtained on the two dimensional torus $\mathbb{T}^2$.

Theorem. *There exists $f \in \mathrm{int}\mathcal{L}WS(\mathbb{T}^2)$ satisfying both Axiom A and the no-cycle condition but not the strong transversality condition.*

The map f is already treated in [3], and the above result will be proved by investigating the dynamics of the map little more. By the theorem, even though a

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diffeomorphism is contained in the C^1-interior of the set of diffeomorphisms possessing the limit weak shadowing property, it does not necessarily satisfy the strong transversality condition.

Let M be as before, and denote by d the distance on M induced from a Riemannian metric on TM. For $f \in \mathrm{Diff}(M)$, denote by $\mathcal{O}_f(x)$ the f-orbit $\{f^i(x)\}_{i\in\mathbb{Z}}$ of $x \in M$. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in M is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z}$. Given $\varepsilon > 0$, $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *ε-shadowed by* $y \in M$ if $d(f^i(y), x_i) < \varepsilon$ for $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-shadowed by some point. It is proved that if f is in the C^1-interior of the set of diffeomorphisms on M satisfying the shadowing property, then f satisfies both Axiom A and the strong transversality condition (see [7]).

Given $\varepsilon > 0$, $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *weakly ε-shadowed* by $y \in M$ if $d(\mathcal{O}_f(y), x_i) < \varepsilon$ for $i \in \mathbb{Z}$. We say that f has the (usual) *weak shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit of f can be weakly ε-shadowed by some point. It is proved in [8] that if f is in the C^1-interior of the set of diffeomorphisms on the closed C^∞ surface satisfying the weak shadowing property, then f satisfies both Axiom A and the no-cycle condition (see [9] for further results). As stated before, if f has the shadowing property, then f has the weak shadowing property.

We say that f has the *limit weak shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}}$, there exists $y \in M$ weakly ε-shadowing $\{x_i\}_{i\in\mathbb{Z}}$, and, if in addition,

$$d(f(x_i), x_{i+1}) \to 0 \quad \text{as} \quad i \to \infty,$$

then

$$d(\mathcal{O}_f(y), x_i) \to 0 \quad \text{as} \quad i \to \infty.$$

Obviously, the limit weak shadowing property is stronger than the usual weak shadowing property by definition, and it is easy to see that every topologically transitive system has the limit weak shadowing property.

The following example constructed by Plamenevskaya [4] gives us useful information about the both weak and limit weak shadowing property.

Example. Represent $\mathbb{T}^2$ as the square $[-2,2] \times [-2,2]$ with identified opposite sides. Let $g : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism with the following properties:

(1) the nonwandering set $\Omega(g)$ of g is the union of 4 hyperbolic fixed points; that is, $\Omega(g) = \{p_1, p_2, p_3, p_4\}$, where p_1 is a source, p_4 is a sink, and p_2, p_3 are saddles;

(2) with respect to coordinates $(v, w) \in [-2,2] \times [-2,2]$, the following conditions hold:

(2.1) $p_1 = (1,2),\ p_2 = (1,0),\ p_3 = (-1,0),\ p_4 = (-1,2)$,

(2.2) $W^u(p_2) \cup \{p_3\} = W^s(p_3) \cup \{p_2\} = [-2,2] \times \{0\}$,

$$W^s(p_2) = \{1\} \times (-2,2),\ W^u(p_3) = \{-1\} \times (-2,2),$$

where $W^s(p_i)$ and $W^u(p_i)$ are the stable and unstable manifolds, respectively, defined as usual;

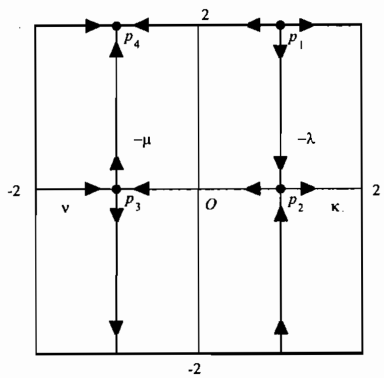
(2.3) there exist neighborhoods U_2, U_3 of p_2, p_3 such that

$$g(x) = p_i + D_{p_i}g(x - p_i) \quad \text{if} \quad x \in U_i,$$

(2.4) there exists a neighborhood U of the point $z = (0,0)$ such that $g(U) \subset U_3$, $g^{-1}(U) \subset U_2$ and g^{-1} is affine on $g(U)$,

(2.5) the eigenvalues of $D_{p_3}g$ are $-\mu$, ν with $\mu > 1$, $0 < \nu < 1$, and the eigenvalues of $D_{p_2}g$ are $-\lambda$, κ with $0 < \lambda < 1$, $\kappa > 1$.

It was proved in [4] that g has the weak shadowing property if and only if the number $\log\lambda/\log\mu$ is irrational. Note that g satisfies Axiom A and the no-cycle condition but does not have the shadowing property.



Now, we show that the map g constructed above (with $\log\lambda/\log\mu$ irrational) does not have the limit weak shadowing property.

For any $\varepsilon > 0$, let $\delta > 0$ be the number of the weak shadowing property of g. Fix two points $x_0 = (1, \delta/4)$ and $x_1 = (1 - \delta/4, 0)$, and set $x_{-i} = g^{-i}(x_0)$ and $x_{i+1} = g^i(x_1)$ for all $i \geq 1$. Then $\{x_i\}_{i=-\infty}^{\infty}$ is a δ-pseudo-orbit of g and

$$\max\{d(x_i, p_3),\ d(x_{-i}, p_1)\} \to 0 \quad \text{as} \quad i \to \infty.$$

Thus, $d(g(x_i), x_{i+1}) \to 0$ as $i \to \infty$.

Let $y \in \mathbb{T}^2$ be any point weakly shadowing $\{x_i\}_{i=-\infty}^{\infty}$. Then $y \notin [-1,1] \times \{0\}$ so that the forward g-orbit of y leaves a neighborhood of p_3 as i increase. Hence the assertion $d(\mathcal{O}_g(y), x_i) \to 0$ as $i \to \infty$ does not hold.

Proof of Theorem.

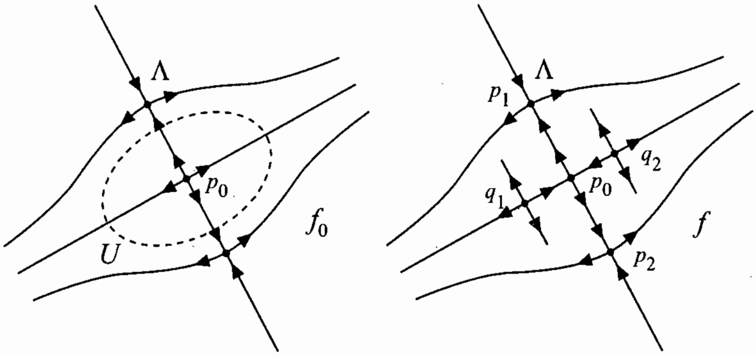
Let $f : X \to X$ be a continuous map on a metric space X. We say that f is *transitive* on an invariant set Y ($\subset X$) provided the forward orbit of some point in Y is dense (see [5, p.39]). Let Y be compact, and let $f : Y \to Y$ be transitive. Then, there exists a residual set $\mathcal{R}^+$ ($\subset Y$) such that for every $y \in \mathcal{R}^+$, the forward orbit, $\mathcal{O}_f^+(y)$, of y is dense (see [5, p.273, Theorem 2.1]). Remark that if $f : Y \to Y$ is transitive, then f has the weak shadowing property.

Let $f \in \text{Diff}(M)$ satisfy Axiom A, and let Λ be a basic set of f. We say that Λ is an *attractor* if it has small neighborhood U (of Λ) with $f(U) \subset U$.

Lemma 1 ([1, p.98]). *Let Λ be a hyperbolic attractor for $f \in \mathrm{Diff}(M)$ satisfying Axiom A. Then $W^s_\varepsilon(\Lambda) = \cup_{x\in\Lambda} W^s_\varepsilon(x)$ is a neighborhood of Λ. Here $W^s_\varepsilon(x)$ ($\varepsilon > 0$) is the local stable manifold of x defined as usual.*

Step 1. *The construction of $f \in \mathrm{Diff}(\mathbb{T}^2)$ satisfying Axiom A but not the strong transversality condition.*

Let f_0 be Smale's *DA*-diffeomorphism on $\mathbb{T}^2$; that is, the non-wandering set is $\Omega(f_0) = \{p_0\} \cup \Lambda$, where p_0 is a fixed point source and Λ is a non-trivial hyperbolic attractor (see [5, p.334]). The map is transitive on Λ and the periodic points of $f_{0|\Lambda}$ are dense, and also it is well-known that f_0 satisfies both Axiom A and the strong transversality condition.



Let U be a open neighborhood of p_0. Then, $\Lambda = \cap_{n=0}^{\infty} f_0^n(\mathbb{T}^2 \setminus U)$ by construction. Now, following [11] (see also [5, p.464]) we modify f_0 to a diffeomorphism f satisfying

- $f_{0|\mathbb{T}^2\setminus U} = f_{|\mathbb{T}^2\setminus U}$,
- there are two fixed point sources q_1 and q_2 and a fixed point saddle p_0 inside U,
- there are two fixed point saddles $\{p_1, p_2\} \subset \Lambda$ such that

$$W^u(p_0, f) \subset W^s(p_1, f) \cup W^s(p_2, f).$$

Here $W^\sigma(p, f)$ ($\sigma = s, u$) is the stable and the unstable manifold of p with respect to f.

Remark that Λ is also a hyperbolic attractor for f, and there are two saddle-connections between a fixed point saddle p_0 and $\{p_1, p_2\} \subset \Lambda$. Hence, f satisfies both Axiom A and the no-cycle condition but does not satisfy the strong transversality condition.

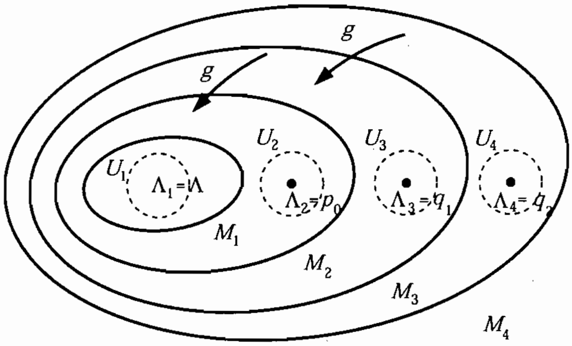
Step 2. *The proof of $f \in \mathrm{int}\mathcal{L}WS(\mathbb{T}^2)$.*

As stated, f satisfies both Axiom A and the no-cycle condition, and so Ω-stable (see [10]). Take a small C^1-neighborhood $\mathcal{U}(f)$ with the Ω-stability of f, and fix any $g \in \mathcal{U}(f)$. Let us show $g \in \mathcal{L}WS(\mathbb{T}^2)$. Of course, g satisfies Axiom A (and the no-cycle condition), and so if g satisfies the strong transversality condition, then, it is well known that g has the shadowing property. Hence, in this case, g has the limit weak shadowing property (cf. [2]). In the following proof, however, it does not matter whether g satisfies the strong transversality condition or not.

It is well-known that there exists a *filtration* with respect to the spectral decomposition $\Omega(g) = \{\Lambda,\ p_0,\ q_1,\ q_2\}$ for g (each element of the set is, of course the continuation of the spectral decomposition for f). For convenience, denote Λ, p_0, q_1, and q_2 by Λ_1, Λ_2, Λ_3, and Λ_4, respectively. By [10], there is a sequence of compact sets

$$\emptyset = M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = \mathbb{T}^2$$

such that $g(M_i) \subset \text{int}M_i$ and $\cap_{m\in\mathbb{Z}} g^m(M_i \setminus M_{i-1}) = \Lambda_i$ for all i.



Let $U_i\ (\subset M_i)$ be any small compact neighborhood of Λ_i for $i = 1, 2, 3, 4$. Then the next lemma is easily checked by making use of the filtration property (cf. [6, Proposition 2.1]).

Lemma 2. *Under the above notations, there are a small number $\delta_0 > 0$ and an integer $I > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ $(0 < \delta \leq \delta_0)$ of g, we have one of the following possibility;*

(*i*) *there are integers I_4 and I_2 $(0 < I_2 - I_4 \leq I)$ such that $x_i \in U_4$ for all $i \leq I_4$, and $x_i \in U_2$ for all $i \geq I_2$,*
(*ii*) *there are integers I_3 and I_2 $(0 < I_2 - I_3 \leq I)$ such that $x_i \in U_3$ for all $i \leq I_3$, and $x_i \in U_2$ for all $i \geq I_2$,*
(*iii*) *there are integers I_4 and I_1 $(0 < I_1 - I_4 \leq I)$ such that $x_i \in U_4$ for all $i \leq I_4$, and $x_i \in U_1$ for all $i \geq I_1$,*
(*iv*) *there are integers I_3 and I_1 $(0 < I_1 - I_3 \leq I)$ such that $x_i \in U_3$ for all $i \leq I_3$, and $x_i \in U_1$ for all $i \geq I_1$,*
(*v*) *there are integers I_2 and I_1 $(0 < I_1 - I_2 \leq I)$ such that $x_i \in U_2$ for all $i \leq I_2$, and $x_i \in U_1$ for all $i \geq I_1$,*
(*vi*) *there are integers I_4, I_2, I_2' and I_1 $(0 < I_2 - I_4 \leq I,\ 0 < I_2' - I_1 \leq I)$ such that $x_i \in U_4$ for all $i \leq I_4$, $x_i \in U_2$ for all $I_2 \leq i \leq I_2'$, and $x_i \in U_1$ for all $i \geq I_1$,*
(*vii*) *there are integers I_3, I_2, I_2' and I_1 $(0 < I_2 - I_3 \leq I,\ 0 < I_2' - I_1 \leq I)$ such that $x_i \in U_3$ for all $i \leq I_3$, $x_i \in U_2$ for all $I_2 \leq i \leq I_2'$, and $x_i \in U_1$ for all $i \geq I_1$.*

Remark. The integer I and the number δ_0 depend only on $\{U_i\}_{i=1}^4$. Since both q_1 and q_2 are sources and δ_0 is small, there are no δ-pseudo-orbits $(0 < \delta \le \delta_0)$ of g from U_4 to U_3, and vice versa.

Recall that any Λ_i is hyperbolic g-invariant set. Thus we may suppose that $g_{|U_i}$ has the (limit) shadowing property. More precisely, we have the following.

Lemma 3 ([2, Theorems 1.2.2 and 1.4.1]). *Under the above notations, for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

(*i*) *for any δ-pseudo-orbit $\{x_i\}_{i=l}^{k-1} \subset U_2$ $(-\infty < l \le i \le k < \infty)$ of g, there is a point $y \in \mathbb{T}^2$ such that $d(g^i(y), x_i) < \varepsilon$ for $l \le i \le k$,*

(*ii*) *for any δ-pseudo-orbit $\{x_i\}_{i=l}^{\infty} \subset U_1$ or U_2 $(-\infty < l < \infty)$ of g, there is a point $y \in \mathbb{T}^2$ such that $d(g^i(y), x_i) < \varepsilon$ for $i \ge l$, and if in addition, $d(g(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then $d(g^i(y), x_i) \to 0$ as $i \to \infty$,*

(*iii*) *for any δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{k} \subset U_2$, U_3 or U_4 $(-\infty < k < \infty)$ of g, there is a point $y \in \mathbb{T}^2$ such that $d(g^i(y), x_i) < \varepsilon$ for $i \le k$.*

Proof of the limit weak shadowing property for g. Fix any $\varepsilon > 0$. Then, by Lemma 1, $W^s_{\varepsilon/2}(\Lambda_1) = \cup_{x \in \Lambda_1} W^s_{\varepsilon/2}(x, g)$ is a neighborhood of Λ_1. Set $U_1 = W^s_{\varepsilon/2}(\Lambda_1)$, and let U_2, U_3 and U_4 be small enough. Let I and δ_0 be as in Lemma 2. We may suppose that if $x \in U_1$ and $d(g(x), y) < \delta_0$, then $g(x), y \in U_1$, and that if $g(x) \in U_j$ and $d(g(x), y) < \delta_0$, then $x, g^{-1}(y) \in U_j$ for $j = 3, 4$.

For any δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ $(0 < \delta \le \delta_0)$ of g, if we have one of the assertions (i) and (ii) of Lemma 2, then $\{x_i\}_{i=-\infty}^{\infty}$ is ε-shadowed by some point because g satisfies the strong transversality condition in some domain where $\{x_i\}_{i=-\infty}^{\infty}$ is included. More accurately, we have the following.

Claim 1. For $\varepsilon > 0$, there exists $0 < \delta_1 \le \delta_0$ such that if a given δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ $(0 < \delta \le \delta_1)$ satisfies one of the assertions (i) and (ii) of Lemma 2, then $\{x_i\}_{i=-\infty}^{\infty}$ is ε-shadowed by some point y. Furthermore, if in addition, $d(g(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then $d(g^i(y), x_i) \to 0$ as $i \to \infty$.

Proof. By the choice of U_j, $g^{-1}(U_j) \subset U_j$ for $j = 3, 4$. Since $0 < \varepsilon \le \delta_0$ is small, there exists $0 < \lambda < 1$ such that if $d(v, w) < \varepsilon$ $(v, w \in U_\varepsilon(U_j))$, then $g^{-1}(v), g^{-1}(w) \in U_\varepsilon(U_j)$ and

$$d(g^{-1}(v), g^{-1}(w)) \le \lambda d(v, w). \tag{1}$$

Next, we pick $0 < \delta' \le \delta_0$ such that if $\{x_i\}_{i=0}^{I}$ is a δ'-pseudo-orbit of g, then $d(g^{-i}(x_I), x_{I-i}) < \varepsilon/2$ for $0 \le i \le I$. Finally we choose $0 < \varepsilon' \le \varepsilon/2$ such that if $d(v, w) < \varepsilon'$ $(v, w \in \mathbb{T}^2)$, then $d(g^{-i}(v), g^{-i}(w)) < \varepsilon/2$ for $0 \le i \le I$.

We deal with only the case $\{x_i\}_{i=-\infty}^{\infty}$ satisfies (i) (other case is similar). By Lemma 3, there exists $0 < \delta_1 = \delta(\varepsilon') \le \delta'$ as in the shadowing property of $g_{|U_2}$. Let $\{x_i\}_{i=-\infty}^{\infty}$ be a given δ-pseudo-orbit $(0 < \delta \le \min\{\delta_1, (1-\lambda)\varepsilon/\lambda\})$ satisfying (i). Since $x_i \in U_2$ for all $i \ge I_2$ and since $g_{|U_2}$ has the shadowing property, there is $y \in \mathbb{T}^2$ such that $d(g^i(y), x_{I_2+i}) < \varepsilon' < \varepsilon$ for all $i \ge 0$. Thus, by Lemma 3, if in addition, $d(g(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then $d(g^i(y), x_i) \to 0$ as $i \to \infty$.

Now, clearly, $d(y, x_{I_2}) < \varepsilon'$, so that $d(g^{-i}(y), g^{-i}(x_{I_2})) < \varepsilon/2$ for all $0 \le i \le I$ by the choice of ε'. On the other hand, by the choice of δ_1, we see $d(g^{-i}(x_{I_2}), x_{I_2-i}) < \varepsilon/2$ for $0 \le i \le I$. Thus, for $0 \le i \le I$,

$$d(g^{-i}(y), x_{I_2-i}) \le d(g^{-i}(y), g^{-i}(x_{I_2})) + d(g^{-i}(x_{I_2}), x_{I_2-i}) < \varepsilon.$$

Obviously, $d(g^{-I}(y), x_{I_2-I}) < \varepsilon$ and $g^{-I}(y), x_{I_2-I} \in U_\varepsilon(U_4)$. Hence, by the choice of ε and by (1)

$$\begin{aligned} d(g^{-I-1}(y), x_{I_2-I-1}) &\le d(g^{-I-1}(y), g^{-1}(x_{I_2-I})) + d(g^{-1}(x_{I_2-I}), x_{I_2-I-1}) \\ &\le d(g^{-1}(g^{-I}(y)), g^{-1}(x_{I_2-I})) \\ &\qquad + d(g^{-1}(x_{I_2-I}), g^{-1}(g(x_{I_2-I-1}))) \\ &< \lambda(\varepsilon + \delta). \end{aligned}$$

By the same way, we see

$$\begin{aligned} d(g^{-I-1}(y), g(x_{I_2-I-2})) &\le d(g^{-I-1}(y), x_{I_2-I-1}) + d(g(x_{I_2-I-2}), x_{I_2-I-1}) \\ &< \lambda(\varepsilon + \delta) + \delta. \end{aligned}$$

Thus $d(g^{-I-2}(y), x_{I_2-I-2}) < \lambda\big(\lambda(\varepsilon + \delta) + \delta\big)$. Hence

$$\begin{aligned} d(g^{-I-2}(y), g(x_{I_2-I-3})) &\le d(g^{-I-2}(y), x_{I_2-I-2}) + d(g(x_{I_2-I-3}), x_{I_2-I-2}) \\ &< \lambda\big(\lambda(\varepsilon + \delta) + \delta\big) + \delta. \end{aligned}$$

Therefore $d(g^{-I-3}(y), x_{I_2-I-3}) < \lambda\big(\lambda\big(\lambda(\varepsilon+\delta)+\delta\big)+\delta\big)$. Repeating the process, we have $d(g^{-I-i}(y), x_{I_2-I-i}) < \varepsilon$ for all $i \ge 0$. ∎

In case, the pseudo-orbit of g satisfies (v), then we have the following ((iii) and (iv) are treated similarly).

Claim 2. For $\varepsilon > 0$, there exists $0 < \delta_2 \le \delta_0$ such that if a given δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ $(0 < \delta \le \delta_2)$ of g satisfies the assertion (v) of Lemma 2, then $\{x_i\}_{i=-\infty}^{I_1}$ is ε-shadowed by some point y.

Proof. There exists $0 < \delta' \le \delta_0$ such that if $\{x_i\}_{i=0}^{I}$ is a δ'-pseudo-orbit of g, then $d(g^i(x_0), x_i) < \varepsilon/2$ for $0 \le i \le I$. Choose $0 < \varepsilon' \le \varepsilon/2$ such that if $d(v, w) < \varepsilon'$ $(v, w \in \mathbb{T}^2)$, then $d(g^i(v), g^i(w)) < \varepsilon/2$ for $0 \le i \le I$. Let $0 < \delta_2 = \delta(\varepsilon') \le \delta'$ be as in the shadowing property of $g_{|U_2}$.

Now, let $\{x_i\}_{i=-\infty}^{\infty}$ be a given δ-pseudo-orbit $(\delta \le \delta_2)$ of g satisfying (v). Since $x_i \in U_2$ for all $i \le I_2$, there is $y \in \mathbb{T}^2$ such that for all $i \ge 0$

$$d(g^{-i}(y), x_{I_2-i}) < \varepsilon' < \varepsilon. \tag{2}$$

Clearly, $d(y, x_{I_2}) < \varepsilon'$, so that $d(g^i(y), g^i(x_{I_2})) < \varepsilon/2$ for all $0 \le i \le I_2$ by the choice of ε'. On the other hand, by the choice of δ', we see $d(g^i(x_{I_2}), x_{I_2+i}) < \varepsilon/2$ for $0 \le i \le I_1 - I_2$. Thus, for $0 \le i \le I_1 - I_2$,

$$d(g^i(y), x_{I_2+i}) \le d(g^i(y), g^i(x_{I_2})) + d(g^i(x_{I_2}), x_{I_2+i}) < \varepsilon. \tag{3}$$

Hence by (2), (3), $\{x_i\}_{i=-\infty}^{I_1}$ is ε-shadowed by some point. ∎

Finally, if the pseudo-orbit of g satisfies one of (vi) and (vii) of Lemma 2, then it is not so hard to show the following claim by combining the above two arguments.

Claim 3. For $\varepsilon > 0$, there exists $0 < \delta_3 \le \delta_0$ such that if a given δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ $(0 < \delta \le \delta_3)$ of g satisfies one of the assertions (vi) and (vii) of Lemma 2, then $\{x_i\}_{i=-\infty}^{I_1}$ is ε-shadowed by some point y.

Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and let $\{x_i\}_{i=-\infty}^{\infty}$ be any δ-pseudo-orbit of g. If the pseudo-orbit satisfies one of the assertions (i) and (ii), then which is ε-shadowed. Suppose that we have one of the assertions (iii), (iv), (v), (vi) and (vii) holds for the pseudo-orbit. By the sequence of claims, we can find $y \in \mathbb{T}^2$ such that

$$d(g^{-i}(y), x_{I_1-i}) < \varepsilon \quad \text{for all} \quad i \ge 0. \tag{4}$$

Thus $\{x_i\}_{i=-\infty}^{I_1} \subset B_\varepsilon(\mathcal{O}_g^-(y))$. Here $\mathcal{O}_g^-(y)$ is the backward g-orbit of y. Remark that $i \ge I_1$ implies $x_i \in U_1 = W^s_{\varepsilon/2}(\Lambda_1)$, and recall that $\Lambda_2 = p_0$, $\Lambda_3 = q_1$ and $\Lambda_4 = q_2$.

Case 1. $y \in W^u(\Lambda_2, g)$.

Notice that Λ_2 is fixed saddle. Take $N > 0$ large enough such that

$$g^{-n}(y) \in W^u_{\varepsilon/2}(\Lambda_2, g) \quad \text{for all} \quad n \ge N. \tag{5}$$

Then, there exists $0 < \nu < \varepsilon/2$ such that if $d(y, z) < \nu$ $(z \in \mathbb{T}^2)$, then

$$d(g^{-n}(y), g^{-n}(z)) < \varepsilon/2 \quad \text{for all} \quad 0 \le n \le N. \tag{6}$$

Hence $\mathcal{O}_g^-(y) \subset U_{\varepsilon/2}(\{z, g^{-1}(z), \cdots, g^{-N}(z)\})$ by (5), (6). Thus, by (4) we have

$$\{x_i\}_{i=-\infty}^{I_1} \subset U_{2\varepsilon}(\{z, g^{-1}(z), \cdots, g^{-N}(z)\} \subset U_{2\varepsilon}(\mathcal{O}_g^-(z))$$

for all $z \in \mathbb{T}$ with $d(z, y) < \nu$.

Case 2. $y \notin W^u(\Lambda_2, g)$.

In this case, we see $y \in W^u(\Lambda_3, g) \cup W^u(\Lambda_4, g)$. Recall that $W^u_{\varepsilon/2}(\Lambda_j, g)$ $(j = 3, 4)$ is 2-dimensional disk since Λ_j is fixed point source, so that if $z \in W^u_{\varepsilon/2}(\Lambda_j, g)$, then $g^{-n}(z) \in W^u_{\varepsilon/2}(\Lambda_j, g)$ for all $n \ge 0$.

We deal with only the case $y \in W^u(\Lambda_3, g)$ (other case is similar). Let $N > 0$ be large enough with $g^{-n}(y) \in W^u_{\varepsilon/2}(\Lambda_3, g)$ for all $n \ge N$. Then, there is $0 < \nu < \varepsilon/2$ such that if $d(y, z) < \nu$ $(z \in \mathbb{T}^2)$, then $d(g^{-n}(y), g^{-n}(z)) < \varepsilon$ for all $n \ge 0$. Hence $\{x_i\}_{i=-\infty}^{I_1} \subset U_{2\varepsilon}(\mathcal{O}_g^-(z))$. In both cases, there exists $\nu > 0$ such that if $d(y, z) < \nu$ $(z \in \mathbb{T}^2)$, then

$$\{x_i\}_{i=-\infty}^{I_1} \subset U_{2\varepsilon}(\mathcal{O}_g^-(z)). \tag{7}$$

Finally, since there exists a residual set $\mathcal{R}^+ \subset \Lambda_1$ such that for every $w \in \mathcal{R}^+$, the forward orbit $\mathcal{O}_g^+(w)$ is dense in Λ_1, we can find $w \in \mathcal{R}^+$ such that $B_{\nu/2}(y) \cap W^s_{\varepsilon/2}(w, g) \ne \emptyset$ (recall that $W^s_{\varepsilon/2}(\Lambda_1) = \cup_{x \in \Lambda_1} W^s_{\varepsilon/2}(x, g)$ is a neighborhood of Λ_1). Thus, if we take a point $z \in B_\nu(y) \cap W^s_{\varepsilon/2}(w, g)$, then $\{x_i\}_{i=-\infty}^{\infty} \subset B_{2\varepsilon}(\mathcal{O}_g(z))$. Indeed, for all $i \ge I_1$, there is $y_i \in \Lambda_1$ such that $x_i \in W^s_{\varepsilon/2}(y_i, g)$ since $W^s_{\varepsilon/2}(\Lambda_1)$ is a neighborhood of Λ_1. On the other hand, since $\mathcal{O}_g^+(w)$ is dense in Λ_1, for all $i \geq I_1$, there exists $n_i > 0$ such that $d(g^{n_i}(w), y_i) < \varepsilon/2$. Clearly, we have $d(g^{n_i}(z), g^{n_i}(w)) < \varepsilon/2$ since $z \in W^s_{\varepsilon/2}(w, g)$. Thus

$$d(g^{n_i}(z), x_i) \leq d(g^{n_i}(z), g^{n_i}(w)) + d(g^{n_i}(w), y_i) < \varepsilon$$

for all $i \geq I_1$. Thus $\{x_i\}_{i=I_1}^{\infty} \subset U_{2\varepsilon}(\mathcal{O}_g^+(z))$, so that $\{x_i\}_{i=-\infty}^{\infty} \subset U_{2\varepsilon}(\mathcal{O}_g(z))$ is concluded from (7).

It only remains to show that if $d(g(x_i), x_{i+1}) \to 0$ as $i \to \infty$, then $d(\mathcal{O}_g^+(z), x_i) \to 0$ as $i \to \infty$. Now, suppose that $d(g(x_i), x_{i+1}) \to 0$ as $i \to \infty$. Recall that for any $\eta > 0$, the stable set $W^s_{\eta/2}(\Lambda_1, g)$ of Λ_1 is a neighborhood of Λ_1. Hence, mimicking the proof of [6, Proposition 2.1] we can see that for any $\eta > 0$, there exists $I_\eta > 0$ such that $\{x_i\}_{i=I_\eta}^{\infty} \subset W^s_{\eta/2}(\Lambda_1, g)$. Thus, for each $i \geq I_\eta$, there exists $y_i \in \Lambda_1$ such that $x_i \in W^s_{\eta/2}(y_i, g)$.

Since $\mathcal{O}_g^+(w)$ is dense in Λ_1 and $z \in W^s_{\varepsilon/2}(w, g)$, for each $i \geq I_\eta$ there exists $m_i > 0$ such that

$$\max\{d(g^{m_i}(w), y_i),\ d(g^{m_i}(z), g^{m_i}(w))\} < \eta/2.$$

Thus we have $d(g^{m_i}(z), x_i) < \eta$ for all $i \geq I_\eta$. ∎

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