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REAL BOTT MANIFOLDS

MIKIYA MASUDA

A fundamental result in the theory of toric varieties says that the categories of toric varieties (over the complex numbers $\mathbb{C}$) and fans are equivalent (see [14]). This reduces the classification of toric varieties to that of fans. Among toric varieties, compact smooth toric varieties which we call toric manifolds are well studied and the classification as varieties is completed for some classes of toric manifolds (see [9], [14], [15] for example).

However, not much is known for the topological classification of toric manifolds, and the following problem is addressed in [12] (see also [11]) and partial affirmative solutions can be found in [5], [11], [12].

Cohomological rigidity problem for toric manifolds. Are two toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings with integer coefficients are isomorphic as graded rings?

The set $X(\mathbb{R})$ of real points in a toric manifold X is called a real toric manifold. It appears as the fixed point set of the complex conjugation on X. For example, when X is a complex projective space $\mathbb{C}P^n$, $X(\mathbb{R})$ is a real projective space $\mathbb{R}P^n$. It is known that

$$H^*(X(\mathbb{R});\mathbb{Z}/2) \cong H^{2*}(X;\mathbb{Z}) \otimes \mathbb{Z}/2$$

for any toric manifold X where $\mathbb{Z}$ denotes the integers and $\mathbb{Z}/2 = \{0, 1\}$, and one may ask the same question as the rigidity problem above for real toric manifolds with $\mathbb{Z}/2$ coefficients, namely

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Cohomological rigidity problem for real toric manifolds. Are two real toric manifolds diffeomorphic (or homeomorphic) if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings?

We are concerned with a sequence of $\mathbb{R}P^1$ bundles

$$M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} M_0 = \{\text{a point}\}$$

such that $M_i \to M_{i-1}$ for $i = 1, \dots, n$ is the projective bundle of a Whitney sum of two real line bundles over M_{i-1}, where one of the two line bundles may be assumed to be trivial without loss of generality *. Grossberg-Karshon [7] considered the sequence above in the complex case and named it a *Bott tower* of height n. Following them, we call the sequence above a *real Bott tower* of height n. The top manifold M_n of a real Bott tower is a real toric manifold. We call it a *real Bott manifold*. The following answers the cohomological rigidity problem affirmatively for real Bott manifolds.

Theorem 1 ([8]). *Two real Bott manifolds are diffeomorphic if their cohomology rings with $\mathbb{Z}/2$ coefficients are isomorphic as graded rings.*

In fact, one can say more than Theorem 1.

Theorem 2 ([10]). *Any graded ring isomorphism between cohomology rings of two real Bott manifolds with $\mathbb{Z}/2$ coefficients is realized by a diffeomorphism between the real Bott manifolds.*

Since $H^1(M_k;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^k$, there are 2^k isomorphism classes in line bundles over M_k; so the number of real Bott towers of height n is $2^{n(n-1)/2}$. The number H_n of diffeomorphism classes in real Bott manifolds of dimension n is unknown for $n \geq 6$ while

$$H_1 = 1,\ H_2 = 2,\ H_3 = 4,\ H_4 = 12,\ H_5 = 54$$

and $H_n > 2^{(n-2)(n-3)/2}$ for any n.

We say that a real Bott manifold is indecomposable if it is not a product of two real Bott manifolds (of positive dimension). Using Theorem 1, one can prove

Theorem 3 ([10]). *The decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique up to permutations of the indecomposable factors. In particular, if $S^1 \times M$ and $S^1 \times M'$ are diffeomorphic for real Bott manifolds M and M', then M and M' are diffeomorphic.*

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*A different sequence of S^1 bundles is considered in [2] where a non-formal symplectic manifold is constructed.

Real toric manifolds provide many examples of aspherical manifolds and real Bott manifolds are examples of flat riemannian manifolds. In fact, any real toric manifold of dimension n supports an action of an elementary abelian 2-group $(\mathbb{Z}/2)^n$ of rank n and real Bott manifolds of dimension n admit a flat riemannian metric invariant under the action of $(\mathbb{Z}/2)^n$. The following shows that these are the only examples among real toric manifolds.

Theorem 4 ([8])**.** *A real toric manifold of dimension n which admits a flat riemannian metric invariant under the action of $(\mathbb{Z}/2)^n$ is a real Bott manifold.*

Finally we remark that a real Bott manifold of dimension n is a small cover over an n-cube and the converse is known to be true up to homeomorphism.

Theorem 5 ([11], [4])**.** *A small cover over an n-cube is homeomorphic to a real Bott manifold of dimension n.*

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Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan.

E-mail address: masuda@sci.osaka-cu.ac.jp