Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 10, Number 2, 2008, pages 1–26

2008 International Workshop on Dynamical Systems and Related Topics

CHAOS, SHADOWING AND HOMOCLINIC ORBITS

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Abstract. The subject of this article is discrete dynamical systems or, more precisely, diffeomorphisms in $\mathbb{R}^n$. We describe the notion of a hyperbolic set, the most important property of which is the shadowing property. We give a proof of the shadowing theorem. Then we show how shadowing can be used to prove that chaos occurs near a transversal homoclinic orbit. Finally we show that shadowing can be used to give computer-assisted proofs of the existence of such orbits. This is a report of my own published work, alone or in collaboration with B.A.Coomes and H.Koçak. However the proof of the Shadowing Theorem (Version 2) given below is new.

1. Hyperbolic Sets and Shadowing

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a diffeomorphism. First we give the definition of hyperbolic set.

Definition 1. A set $S \subset \mathbb{R}^n$ is said to be a *hyperbolic set* if it is *invariant*, that is, $f(S) = S$, and there is a splitting

$$\mathbb{R}^n = E^s(x) \oplus E^u(x),\ x \in S$$

such that the subspaces $E^s(x)$ and $E^u(x)$ have constant dimensions and have the invariance properties

$$Df(x)(E^s(x)) = E^s(f(x)), \quad Df(x)(E^u(x)) = E^u(f(x)), \quad x \in S$$

2000 *Mathematics Subject Classification.* 58F22, 58F13, 65H10.
Key words and phrases. Hyperbolic sets, transversal homoclinic orbits, pseudo orbits, chaos, shadowing, Newton's method.

and there are positive constants K and $\lambda < 1$ such that for $k \geq 0$ and $x \in S$

$$\|Df^k(x)\xi\| \leq K\lambda^k\|\xi\| \quad \text{for } \xi \in E^s(x) \quad \text{and} \quad \|Df^{-k}(x)\xi\| \leq K\lambda^k\|\xi\| \quad \text{for } \xi \in E^u(x).$$

Next we define the notions of pseudo orbit and shadowing.

Definition 2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism. A sequence $\{y_k\}_{k=-\infty}^{\infty}$ of points is said to be a δ *pseudo orbit* of f if

$$\|y_{k+1} - f(y_k)\| \leq \delta \text{ for } k \in \mathbb{Z}.$$

Definition 3. An orbit $\{x_k\}_{k=-\infty}^{\infty}$ of f, that is, $x_{k+1} = f(x_k)$ for all k, is said to ε-*shadow* the δ pseudo orbit $\{y_k\}_{k=-\infty}^{\infty}$ if

$$\|x_k - y_k\| \leq \varepsilon \text{ for } k \in \mathbb{Z}.$$

The Shadowing Theorem (Version 1): Now we state our first version of the shadowing theorem in which we consider a single pseudo orbit.

Theorem 1. *Let $\{y_k\}_{k=-\infty}^{+\infty}$ be a bounded δ pseudo orbit of a C^1 diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$.*
Let $L\colon \ell^\infty(\mathbb{Z}, \mathbb{R}^n) \to \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ be the linear operator defined for $\mathbf{u} = \{u_k\}_{k=-\infty}^{+\infty}$ by

$$(L\mathbf{u})_k = u_{k+1} - Df(y_k)u_k \quad \text{for } k \in \mathbb{Z}.$$

Suppose that L is invertible, set

$$\varepsilon = 2\|L^{-1}\|\delta$$

and define the modulus of continuity

$$\omega(\varepsilon) = \sup\big\{\|Df(x + y_k) - Df(y_k)\| : \|x\| \leq \varepsilon,\ k \in \mathbb{Z}\big\}.$$

Then if

$$2\|L^{-1}\|\omega(2\|L^{-1}\|\delta) \leq 1,$$

there is a unique true orbit $\{x_k\}_{k=-\infty}^{+\infty}$ of f which ε-shadows $\{y_k\}_{k=-\infty}^{+\infty}$, that is, $x_{k+1} = f(x_k)$ for all k and

$$\|x_k - y_k\| \leq \varepsilon = 2\|L^{-1}\|\delta \quad \text{for } k \in \mathbb{Z}.$$

Proof. We need to find a sequence $\{x_k\}_{k=-\infty}^{+\infty}$ such that for all k,

$$x_{k+1} = f(x_k) \quad \text{and} \quad \|x_k - y_k\| \leq \varepsilon.$$

Write

$$z_k = x_k - y_k.$$

Then for all k, $\|z_k\| \leq \varepsilon$ and

$$z_{k+1} = Df(y_k)z_k + g_k(z_k),$$

where

$$g_k(z) = f(z + y_k) - f(y_k) - Df(y_k)z + f(y_k) - f(y_{k+1}).$$

So we need to find a solution $\mathbf{z} = \{z_k\}_{k=-\infty}^{+\infty}$ in $\ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ of

$$L\mathbf{z} = g(\mathbf{z})$$

such that $\|\mathbf{z}\| \leq \varepsilon$, where

$$[g(\mathbf{z})]_k = g_k(z_k)$$

and

$$\|\mathbf{z}\| = \sup_{k \in \mathbb{Z}} \|z_k\|.$$

We write this equation as

$$\mathbf{z} = T\mathbf{z} = L^{-1}g(\mathbf{z}).$$

Note that

$$\|g(\mathbf{0})\| \leq \delta, \quad Dg(\mathbf{0}) = 0 \quad \text{and} \quad \|Dg(\mathbf{z})\| \leq \omega(\varepsilon) \quad \text{if} \quad \|\mathbf{z}\| \leq \varepsilon.$$

If $\mathbf{z}$ is a sequence with $\|\mathbf{z}\| \leq \varepsilon$, then

$$\|T(\mathbf{z})\| \leq \|L^{-1}\|\|g(\mathbf{z})\| \leq \|L^{-1}\|[\delta + \omega(\varepsilon)\varepsilon] \leq \varepsilon$$

and if $\mathbf{z}$ and $\mathbf{w}$ are sequences with $\|\mathbf{z}\| \leq \varepsilon$, $\|\mathbf{w}\| \leq \varepsilon$,

$$\|T(\mathbf{z}) - T(\mathbf{w})\| \leq \|L^{-1}\|\|g(\mathbf{z}) - g(\mathbf{w})\| \leq \|L^{-1}\|\omega(\varepsilon)\|\mathbf{z} - \mathbf{w}\| \leq \frac{1}{2}\|\mathbf{z} - \mathbf{w}\|.$$

Then the theorem follows using the contraction mapping principle applied to T.

The Shadowing Theorem (Version 2): Now we give the second version of the Shadowing Theorem. This one deals with a hyperbolic set.

Theorem 2. *Let S be a compact hyperbolic set for a C^1 diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Then there exist positive constants δ_0 and M such that any δ pseudo orbit of f in S with $\delta \leq \delta_0$ is ε-shadowed by a unique true hyperbolic orbit of f with*

$$\varepsilon = M\delta.$$

Proof. Let S be a compact hyperbolic set for a C^1 diffeomorphism $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. Suppose U is a bounded convex open set containing S. If $\{y_k\}_{k=-\infty}^{\infty}$ is a δ pseudo orbit in S, then when $k \geq m$

$$\|y_k - f^{k-m}(y_m)\| \leq (1 + M_1 + \ldots + M_1^{k-m-1})\delta, \tag{1}$$

where the right side is interpreted as zero when $k = m$ and

$$M_1 = \sup_{x\in U} \|Df(x)\|.$$

Inequality (1) follows by induction on k using the estimate

$$\begin{aligned} \|y_{k+1} - f^{k+1-m}(y_m)\| &\leq \|y_{k+1} - f(y_k)\| + \|f(y_k) - f(f^{k-m}(y_m))\| \\ &\leq \delta + M_1\|y_k - f^{k-m}(y_m)\|. \end{aligned}$$

We fix a positive integer m to be determined later and define a sequence $\{z_k\}_{k=-\infty}^{+\infty}$ as follows:

$$z_k = f^{k-rm}(y_{rm}) \quad \text{for} \quad rm \leq k < (r+1)m,\ r \in \mathbb{Z}.$$

Then it follows from (1) that for all k

$$\|z_k - y_k\| \leq (1 + M_1 + \ldots + M_1^{m-2})\delta.$$

We now show under certain conditions on δ and m that the linear operator $\tilde{L}\colon \ell^\infty(\mathbb{Z}, \mathbb{R}^n) \to \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ defined by

$$(\tilde{L}\mathbf{u})_k = u_{k+1} - Df(z_k)u_k \quad \text{for } k \in \mathbb{Z} \tag{2}$$

is invertible and also obtain an upper bound for the norm of its inverse. To this end, we define the projections

$$P_k^{(r)} = \mathcal{P}(f^{k-rm}(y_{rm}))$$

for $rm \leq k \leq (r+1)m$ and $r \in \mathbb{Z}$. Here $\mathcal{P}(x)$ is the projection with range $E^s(x)$ and nullspace $E^u(x)$. It is well-known (see, for example, Palmer [2000]) that $\mathcal{P}(x)$ is a continuous function of x. Note, using (1), that

$$\left\|P_{(r+1)m}^{(r)} - P_{(r+1)m}^{(r+1)}\right\| = \|\mathcal{P}(f^m(y_{rm})) - \mathcal{P}(y_{(r+1)m})\| \leq \delta_1 = \omega((1+M_1+\ldots+M_1^{m-1})\delta), \tag{3}$$

where $\omega(\cdot)$ is the modulus of continuity for $\mathcal{P}(\cdot)$ on S.

To show that the operator $\tilde{L}$ has an inverse, we look for a unique bounded solution of the difference equation

$$u_{k+1} = Df(z_k)u_k + g_k,\ k \in \mathbb{Z}, \tag{4}$$

where $\{g_k\}_{k=-\infty}^{+\infty}$ is an arbitrary bounded sequence. On any interval $[rm,(r+1)m]$, for given $\xi_r \in \mathcal{R}(P_{rm}^{(r)})$ and $\eta_{r+1} \in \mathcal{N}\left(P_{(r+1)m}^{(r)}\right)$, (4) has a unique solution $u_k^{(r)}$ satisfying

$$P_{rm}^{(r)} u_{rm}^{(r)} = \xi_r, \quad (I - P_{(r+1)m}^{(r)}) u_{(r+1)m}^{(r)} = \eta_{r+1}.$$

This solution is given by

$$\begin{aligned} u_k^{(r)} = u_k^{(r)}(\xi_r,\eta_{r+1}) &= \Phi^{(r)}(k,rm)\xi_r + \Phi^{(r)}(k,(r+1)m)\eta_{r+1} \\ &\quad + \sum_{\ell=rm}^{k-1} \Phi^{(r)}(k,\ell+1) P_{\ell+1}^{(r)} g_\ell \\ &\quad - \sum_{\ell=k}^{(r+1)m-1} \Phi^{(r)}(k,\ell+1)\left(I - P_{\ell+1}^{(r)}\right) g_\ell, \end{aligned} \tag{5}$$

where

$$\Phi^{(r)}(k,\ell) = Df^{k-\ell}(f^{\ell-rm}(y_{rm}))$$

is the transition matrix for the difference equation

$$u_{k+1} = Df(z_k)u_k, \ k \in \mathbb{Z}$$

on the interval $[rm,(r+1)m]$. Note, in fact, that all solutions of (4) on the interval $[rm,(r+1)m]$ can be represented in the form (5). Note also that $\|\mathcal{P}(x)\|$ is bounded and so we can adjust K so that the inequalities in Definition 1 imply that for $k \ge 0$ and $x \in S$

$$\|Df^k(x)\mathcal{P}(x)\| \le K\lambda^k \quad \text{and} \quad \|Df^{-k}(x)(I-\mathcal{P}(x))| \le K\lambda^k.$$

Now we estimate

$$\begin{aligned} \|u_k^{(r)}\| &\le K\lambda^{k-rm}\|\xi_r\| + K\lambda^{(r+1)m-k}\|\eta_{r+1}\| + \left[\sum_{\ell=rm}^{k-1} K\lambda^{k-\ell-1} + \sum_{\ell=k}^{(r+1)m-1} K\lambda^{\ell+1-k}\right]\|\mathbf{g}\| \\ &\le K[\|\xi_r\| + \|\eta_{r+1}\|] + K\left[(1-\lambda)^{-1}(1-\lambda^{k-rm}) + \lambda(1-\lambda)^{-1}(1-\lambda^{(r+1)m-k})\right]\|\mathbf{g}\| \\ &\le K[\|\xi_r\| + \|\eta_{r+1}\|] + K(1-\lambda)^{-1}(1+\lambda)\|\mathbf{g}\|, \end{aligned} \tag{6}$$

where $\|\mathbf{g}\| = \sup_{k\in\mathbb{Z}} \|g_k\|$ is the norm in $\ell^\infty(\mathbb{Z}, \mathbb{R}^n)$. So we will have a unique bounded solution if and only if there exists a unique bounded sequence $\{(\xi_r,\eta_r)\}_{r=-\infty}^{\infty}$ such that

$$u_{rm}^{(r-1)}(\xi_{r-1},\eta_r) = u_{rm}^{(r)}(\xi_r,\eta_{r+1})$$

for all r. This latter equation can be written as

$$\xi_r - \eta_r - \Phi^{(r-1)}(rm,(r-1)m)\xi_{r-1} + \Phi^{(r)}(rm,(r+1)m)\eta_{r+1} = c_r,$$

where

$$(7)\quad c_r = \sum_{\ell=(r-1)m}^{rm-1} \Phi^{(r-1)}(rm,\ell+1)P^{(r-1)}_{\ell+1} g_\ell + \sum_{\ell=rm}^{(r+1)m-1} \Phi^{(r)}(rm,\ell+1)(I-P^{(r)}_{\ell+1})g_\ell.$$

Now denote by X the Banach space of bounded sequences $\{(\xi_r,\eta_r)\}_{r=-\infty}^{+\infty}$, where $\xi_r \in \mathcal{R}(P^{(r)}_{rm})$, $\eta_r \in \mathcal{N}(P^{(r-1)}_{rm})$, with norm

$$\max\left\{\sup_{r\in\mathbb{Z}} \|\xi_r\|, \sup_{r\in\mathbb{Z}} \|\eta_r\|\right\}.$$

We define the operator $T : X \mapsto \ell^\infty(\mathbb{Z},\mathbb{R}^n)$ by

$$\{(\xi_r,\eta_r)\}_{r=-\infty}^{+\infty} \mapsto \{\xi_r - \eta_r - \Phi^{(r-1)}(rm,(r-1)m)\xi_{r-1} + \Phi^{(r)}(rm,(r+1)m)\eta_{r+1}\}_{r=-\infty}^{+\infty}.$$

Then we need to solve the equation

$$T\{(\xi_r,\eta_r)\}_{r=-\infty}^{+\infty} = \mathbf{c},$$

where $\mathbf{c} = \{c_r\}_{r=-\infty}^{+\infty}$ is in $\ell^\infty(\mathbb{Z},\mathbb{R}^n)$ because

$$(8)\quad \begin{aligned} \|c_r\| &\le \left[\sum_{\ell=(r-1)m}^{rm-1} K\lambda^{rm-\ell-1} + \sum_{\ell=rm}^{(r+1)m-1} K\lambda^{\ell+1-rm}\right]\|\mathbf{g}\| \\ &= K\left[(1-\lambda)^{-1}(1-\lambda^m) + \lambda(1-\lambda)^{-1}(1-\lambda^m)\right]\|\mathbf{g}\| \\ &\le K(1-\lambda)^{-1}(1+\lambda)\|\mathbf{g}\|. \end{aligned}$$

To show that T is invertible, we consider another operator $T_0 : X \mapsto \ell^\infty(\mathbb{Z},\mathbb{R}^n)$ defined by

$$\{(\xi_r,\eta_r)\}_{r=-\infty}^{+\infty} \mapsto \{\xi_r - \eta_r\}_{r=-\infty}^{+\infty}.$$

To show this operator is invertible, we use the following lemma.

Lemma 1. *Let P and Q be projections such that $\|P-Q\| \le \delta < 1$. Then*

$$\mathbb{R}^n = \mathcal{R}(P) \oplus \mathcal{N}(Q)$$

and if R is the projection on to $\mathcal{R}(P)$ along $\mathcal{N}(Q)$, then

$$\|R-P\| \le \delta(1-\delta)^{-1}\|P\|, \quad \|R-Q\| \le \delta(1-\delta)^{-1}\|Q\|.$$

Proof. Suppose $x \in \mathcal{R}(P)\cap\mathcal{N}(Q)$. Then

$$\|x\| = \|(P-Q)x\| \le \|P-Q\|\|x\| \le \delta\|x\|$$

and so $x = 0$. Next suppose x is orthogonal to $\mathcal{R}(P)\oplus\mathcal{N}(Q)$. Then

$$x^*Px = 0, \quad x^*(I-Q)x = 0$$

and so

$$\|x\|^2 = x^*x = x^*Qx = x^*(Q-P)x \le \|Q-P\|\|x\|^2 \le \delta\|x\|^2$$

and so $x = 0$. Thus $\mathbb{R}^n = \mathcal{R}(P) \oplus \mathcal{N}(Q)$.

Next note that

$$RP = R, \quad PR = P, \quad RQ = Q, \quad QR = R.$$

Then

$$\|R-P\| = \|(Q-P)R\| = \|(Q-P)(R-P) + (Q-P)P\| \le \delta\|R-P\| + \delta\|P\|$$

and so

$$\|R-P\| \le \delta(1-\delta)^{-1}\|\|P\|.$$

Similarly,

$$\|R-Q\| = \|R(P-Q)\| = \|(R-Q)(P-Q) + Q(P-Q)\| \le \delta\|R-Q\| + \delta\|Q\|$$

and so

$$\|R-Q\| \le \delta(1-\delta)^{-1}\|Q\|.$$

Thus Lemma 1 is proved.

If the δ_1 in (3) satisfies

$$\delta_1 \le \frac{1}{2}, \tag{9}$$

we may use Lemma 1 to deduce that the solution of the equation (here $\mathbf{c}$ is arbitrary)

$$T_0\{(\xi_r, \eta_r)\}_{r=-\infty}^{+\infty} = \mathbf{c},$$

which written in components is

$$\xi_r - \eta_r = c_r,$$

is given by

$$\xi_r = R_r c_r, \quad \eta_r = -(I - R_r)c_r,$$

where R_r is the projection with range $\mathcal{R}(P_{rm}^{(r)})$ and nullspace $\mathcal{N}(P_{rm}^{(r-1)})$. Also from Lemma 1

$$\|R\| \le \|R_r - P_{rm}^{(r)}\| + \|P_{rm}^{(r)}\| \le [\delta_1(1-\delta_1)^{-1} + 1]\|P_{rm}^{(r)}\| \le [1 + \delta_1(1-\delta_1)^{-1}]K \le 2K.$$

Similarly,

$$\|I - R_r\| \le \|R_r - P_{rm}^{(r)}\| + \|I - P_{rm}^{(r)}\| \le 2K.$$

It follows that T_0 is invertible with

$$\|T_0^{-1}\| \le 2K.$$

Next note that since
$\|\Phi^{(r-1)}(rm,(r-1)m)\xi_{r-1}\| \le K\lambda^m\|\xi_{r-1}\|$ and $\|\Phi^{(r)}(rm,(r+1)m)\eta_{r+1}\|$
$\le K\lambda^m\|\eta_{r+1}\|$, we have

$$\|T - T_0\| \le K\lambda^m.$$

So if

$$2K \cdot K\lambda^m \le \frac{1}{2}, \tag{10}$$

T is invertible with

$$\|T^{-1}\| \le (1 - \|T - T_0\|\|T_0^{-1}\|)^{-1}\|T_0^{-1}\| \le 4K. \tag{11}$$

Now if we take $\{(\xi_r, \eta_r)\}_{r=-\infty}^{+\infty} = T^{-1}\mathbf{c}$, where $\mathbf{c}$ is defined in (7), then $u_k^{(r)}$ as defined in (5) gives the unique bounded solution of (4) and, using (6), (8) and (11) we estimate

$$\|u_k^{(r)}\| \le 2K\|T^{-1}\|\,\|\mathbf{c}\| + K(1-\lambda)^{-1}(1+\lambda)\|\mathbf{g}\| \le K(1+8K^2)(1-\lambda)^{-1}\|\mathbf{g}\|.$$

It follows that the operator $\tilde{L}$ is invertible and

$$\|\tilde{L}^{-1}\| \le K(1+8K^2)(1-\lambda)^{-1}.$$

Finally let L be the operator as defined in Theorem 1. Note that

$$|Df(y_k) - Df(z_k)| \le \delta_2 = \omega_0((1 + M_1 + \ldots + M_1^{m-2})\delta),$$

where ω_0 is the modulus of continuity of Df on S. It follows that

$$\|L - \tilde{L}\| \le \delta_2.$$

Then if

$$K(1+8K^2)(1-\lambda)^{-1}\delta_2 \le \frac{1}{2}, \tag{12}$$

the operator L is also invertible and

$$\|L^{-1}\| \le M/2,$$

where

$$M = 4K(1+8K^2)(1-\lambda)^{-1}.$$

Now choose m as the smallest positive integer such that (10) holds and then choose δ_0 as the smallest δ such that (9), (12) and

$$M\omega_0(M\delta) \le 1$$

hold. Then if $\delta \le \delta_0$, we know that L is invertible with $\|L^{-1}\| \le M/2$ and

$$2\|L^{-1}\|\omega_0(2\|L^{-1}\|\delta) \le 1.$$

Then it follows from Theorem 1 that there is a unique true orbit $\{x_k\}_{k=-\infty}^{+\infty}$ of f which ε-shadows $\{y_k\}_{k=-\infty}^{+\infty}$, where $\varepsilon = 2\|L^{-1}\|\delta \leq M\delta$. This completes the proof of the theorem.

2. Transversal homoclinic points

In this section we use shadowing to show that a diffeomorphism has chaotic behaviour in the neignbourhood of the orbit of a transversal homoclinic point. Then we exhibit some diffeomorphisms which have transversal homoclinic points.

Definition 4. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a diffeomorphism. A point y_0 is said to be a *homoclinic point* with respect to the fixed point x_0 if

$$f^k(y_0) \to x_0 \quad \text{as} \quad k \to \pm\infty.$$

If, in addition, the compact invariant set

$$S = \{x_0\} \cup \{f^k(y_0) : k \in \mathbb{Z}\}$$

is *hyperbolic*, we say the homoclinic point is *transversal.*

Note that transversality is usually defined in terms of the tangent spaces to the stable and unstable manifolds. The definition given above is equivalent and is more suited to our purpose here.

2.1. **Symbolic dynamics near a transversal homoclinic point. Theorem 3 (Poincaré-Birkhoff-Smale).** *Let x_0 be a hyperbolic fixed point of the C^1 diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with associated transversal homoclinic orbit*

$$\{y_k = f^k(y_0)\}_{k=-\infty}^{+\infty}.$$

Then there is a set $\tilde{S}$ near x_0 and a positive integer J such that $f^J(\tilde{S}) = \tilde{S}$ and $f^J : \tilde{S} \mapsto \tilde{S}$ is topologically conjugate to the shift on two symbols, that if, if Σ is the set of doubly infinite sequences $\{e_i\}_{i=-\infty}^{+\infty}$ of 0's and 1's endowed with the discrete topology, there a homeomorphism $h : \Sigma \mapsto \tilde{S}$ such that

$$h \circ \sigma = f^J \circ h,$$

where $\sigma : \Sigma \mapsto \Sigma$ is the shift

$$\sigma(\{e_i\}) = \{e_{i+1}\}.$$

Thus f is chaotic on $\tilde{S}$.

Proof. We choose $m > 0$ so that

$$\|f^{m+1}(y_0) - x_0\| \leq \delta/2, \quad \|f^{-m}(y_0) - x_0\| \leq \delta/2,$$

where $\delta > 0$ is small enough to apply the Shadowing Theorem (Version 2) to the hyperbolic set

$$S = \{x_0\} \cup \{f^k(y_0) : k \in \mathbb{Z}\}$$

and such that $2M\delta < \|x_0 - y_0\|$.

The symbol 0 corresponds to an orbit segment

$$\{x_0, \ldots, x_0\} \tag{13}$$

with $2m + 1$ points and the symbol 1 to an orbit segment

$$\{f^{-m}(y_0), \ldots, y_0, \ldots, f^m(y_0)\}. \tag{14}$$

We construct a δ pseudo orbit by stringing these orbit segments together.

Consider a sequence $e = \{e_i\}_{i=-\infty}^{+\infty}$ of zeros and ones. If $e_i = 0$ we take the orbit segment (13) and if $e_i = 1$ we take the orbit segment (14). By the Shadowing Theorem, there exists a unique true orbit which ε-shadows this δ pseudo orbit with $\varepsilon = M\delta$.

Then we define $h(e)$ as the point on this orbit which shadows the first point in the orbit segment corresponding to e_0. By uniqueness, it follows that

$$h(\sigma(e)) = f^{2m+1}(h(e)).$$

Since Σ is compact Hausdorff, to prove that h is a homeomorphism we need only show that h is one-one and continuous.

Suppose $h(e) = h(\tilde{e})$ but $e \neq \tilde{e}$. Then there exists i such that $e_i \neq \tilde{e}_i$. Without loss of generality, we can suppose $e_i = 1$ and $\tilde{e}_1 = 0$. Let $z_k = f^k(h(e))$. Choose k_0 so that z_{k_0} shadows the midpoints of the segments corresponding to e_i and $\tilde{e}_i$. Then

$$\|z_{k_0} - y_0\| \leq M\delta \quad \text{and} \quad \|z_{k_0} - x_0\| \leq M\delta$$

so that

$$\|x_0 - y_0\| \leq 2M\delta,$$

which contradicts our choice of δ. Hence h is one-one.

To prove h is continuous, we proceed by contradiction. Suppose there exists e and $\varepsilon > 0$ and a sequence $e^{(p)} \to e$ as $p \to \infty$ but for all p

$$\|h(e^{(p)}) - h(e)\| \geq \varepsilon.$$

Write $z_p = h(e^{(p)})$. By compactness, there exists a subsequence $z_{j_p} \to z$. Then

$$\|z - h(e)\| \geq \varepsilon. \tag{15}$$

Since $e^{(j_p)} \to e$, $e_i^{(j_p)} = e_i$ for $|i| \le I$ if p is sufficiently large. Then, if y_k is the pseudo-orbit corresponding to e,

$$\|f^k(z_{j_p}) - y_k\| \le M\delta \quad \text{for} \quad |k| \le (2m+1)I$$

if p is large. Letting $p \to \infty$, it follows that

$$\|f^k(z) - y_k\| \le M\delta \quad \text{for} \quad |k| \le (2m+1)I.$$

This holds for all I. So by uniqueness, $z = h(e)$, which contradicts (15). Hence h is continuous and the theorem follows with $J = 2m+1$.

2.2. **Finding Transversal Homoclinic Points: the Melnikov method.** It is not easy to write down a diffeomorphism with a transversal homoclinic point. The idea of the Melnikov method is to begin with a non-transversal homoclinic point (it is very easy to write down examples of such) and perturb it so it becomes transversal. We state the theorem without proof (see, for example, Palmer [2000] for a proof).

Theorem 4. *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R} \to \mathbb{R}^2$ be C^2 functions such that for a positive number T*

$$h(t+T) \equiv h(t)$$

and the autonomous system

$$\dot{x} = g(x)$$

has a saddle point x_0 with associated homoclinic orbit $\zeta(t)$. Then

(a) *for μ sufficiently small the period map of the system*

$$\dot{x} = g(x) + \mu h(t) \tag{16}$$

has a unique hyperbolic fixed point $x(\mu)$ near x_0;

(b) *if we set*

$$\Delta(\alpha) = \int_{-\infty}^{\infty} e^{-\int_0^t \operatorname{Tr} Dg(\zeta(s+\alpha))ds} \left[g(\zeta(t+\alpha)) \wedge h(t)\right] dt$$

(note if a and b are vectors in $\mathbb{R}^2$, then $a \wedge b = a_1 b_2 - a_2 b_1$) and there exists α_0 such that

$$\Delta(\alpha_0) = 0, \;\; \Delta'(\alpha_0) \ne 0,$$

then for μ sufficiently small but nonzero the period map for (16) has a transversal homoclinic point $y(\mu)$ near $\zeta(\alpha_0)$ associated with the hyperbolic fixed point $x(\mu)$.

Example 1. The second order equation $\ddot{x} + 2x^3 - x = \mu \cos t$ can be written as the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - 2x_1^3 + \mu \cos t.$$

$(0,0)$ is a saddle point for the autonomous system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - 2x_1^3$$

and $(\xi(t), \dot{\xi}(t))$, where $\xi(t) = \operatorname{sech} t$, is an associated homoclinic orbit. The Melnikov function is

$$\Delta(\alpha) = \int_{-\infty}^{\infty} \operatorname{sech} t \ \sin(t - \alpha) dt.$$

We see that $\Delta(0) = 0$ and

$$\Delta'(0) = -\int_{-\infty}^{\infty} \operatorname{sech} t \cos t \, dt \neq 0.$$

Hence the conclusions of the theorem apply with $\alpha_0 = 0$.

Here is a similar kind of result where now the system is slowly varying. The first such results seems to have been proved by Cherry and later Kurland and Levi. We state the theorem without proof. (For more information about these kinds of results see, for example, Battelli and Palmer [2001, 2008].)

Theorem 5. *Suppose $g(x, \alpha)$ is periodic in α and for each fixed α the planar system*

$$\dot{x} = g(x, \alpha)$$

has a saddle point $x = w(\alpha)$ (periodic in α). Suppose next for some α_0, the equation

$$\dot{x} = g(x, \alpha_0)$$

has a solution $x_0(t)$ such that

$$x_0(t) \to w(\alpha_0) \text{ as } |t| \to \infty.$$

Denote by $\psi(t)$ the unique (up to a scalar multiple) nonzero bounded solution of the system adjoint to

$$\dot{x} = g_x(x_0(t), \alpha_0)x.$$

Then if

$$\int_{-\infty}^{\infty} \psi^*(t) g_\alpha(x_0(t), \alpha_0) dt \neq 0, \tag{17}$$

the period map for

$$\dot{x} = g(x, \varepsilon t)$$

has a transversal homoclinic point when $\varepsilon > 0$ is sufficiently small.

Remark 1. Condition (17) implies that the saddle point connexion $x_0(t)$ for $\dot{x} = g(x, \alpha_0)$ breaks as α passes through α_0.

Example 2. The second order equation $\ddot{x} + a(\varepsilon t)\dot{x} + 2x^3 - x = 0$, where $a(\alpha)$ is a periodic function with $a(0) = 0$, $a'(0) \neq 0$, can be written as the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a(\varepsilon t)x_2 - 2x_1^3 + x_1.$$

For all α, $(0, 0)$ is a saddle point for the autonomous system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a(\alpha)x_2 + x_1 - 2x_1^3$$

and, when $\alpha = 0$, $x_0(t) = (\xi(t), \dot{\xi}(t))$, where $\xi(t) = \operatorname{sech} t$, is an associated homoclinic orbit. In this case the integral in (17) is

$$a'(0) \int_{-\infty}^{\infty} \dot{\xi}(t)^2 dt \neq 0.$$

Hence the conclusions of the theorem apply with $\alpha_0 = 0$.

3. Finding Homoclinic Orbits by Numerical Shadowing

In this section we show how shadowing ideas can be used to construct diffeomorphisms with transversal homoclinic points associated with fixed points. We will construct these homoclinic orbits by shadowing pseudo homoclinic orbits, which we now define.

Definition 5. Let x_0 be a hyperbolic fixed point of a C^2 map $f\colon \mathbb{R}^n \to \mathbb{R}^n$. A sequence $\{y_k\}_{k=-\infty}^{+\infty}$ is said to be a *δ pseudo homoclinic orbit with respect to x_0* if

(i) $\|y_{k+1} - f(y_k)\| \leq \delta$ for $k \in \mathbb{Z}$,

(ii) $y_k = x_0$ for $k \leq p$ and $y_k = x_0$ for $k \geq q$ for some integers $p < q$.

For a given bounded δ pseudo homoclinic orbit $\{y_k\}_{k=-\infty}^{+\infty}$, we define the linear operator $L\colon \ell^\infty(\mathbb{Z}, \mathbb{R}^n) \to \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ by

$$(L\mathbf{u})_k = u_{k+1} - Df(y_k)u_k \quad \text{for } k \in \mathbb{Z},$$

where $\mathbf{u} = \{u_k\}_{k=-\infty}^{+\infty} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$. Set

$$M = \sup\{\|D^2 f(x)\| : x \in \mathbb{R}^n\}.$$

Theorem 6. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 diffeomorphism and x_0 is a hyperbolic fixed point of f. Let $\{y_k\}_{k=-\infty}^{+\infty}$ be a δ pseudo homoclinic orbit of f with respect to x_0. Then if L is invertible and*

$$2M\|L^{-1}\|^2\delta < 1,$$

(i) *the pseudo homoclinic orbit* $\{y_k\}_{k=-\infty}^{+\infty}$ *is* ε*-shadowed by a unique true hyperbolic orbit* $\{z_k\}_{k=-\infty}^{+\infty}$*, where*

$$\varepsilon = 2\|L^{-1}\|\delta,$$

that is, for all k, $z_{k+1} = f(z_k)$ *and*

$$\|z_k - y_k\| \leq \varepsilon;$$

(ii) *moreover, if*

$$\|y_k - x_0\| > \varepsilon$$

for some k *with* $p < k < q$*, the point* z_0 *is a transversal homoclinic point with respect to the fixed point* x_0*.*

Proof. The existence of a unique true orbit $\{z_k\}_{k=-\infty}^{+\infty}$ shadowing $\{y_k\}_{k=-\infty}^{+\infty}$ follows from the Shadowing Theorem (Version 1). To prove it is hyperbolic, according to Slyusarchuk [1983], it is enough to show that the linear operator $T\colon \ell^\infty(\mathbb{Z}, \mathbb{R}^n) \to \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ defined by

$$(T\mathbf{u})_k = u_{k+1} - Df(z_k)u_k \quad \text{for } k \in \mathbb{Z}$$

is invertible. Note that

$$\|T - L\| \leq \sup_{k\in\mathbb{Z}} \|Df(z_k) - Df(y_k)\| \leq M\varepsilon$$

so that

$$\|T - L\|\|L^{-1}\| \leq M\varepsilon\|L^{-1}\| = 2M\|L^{-1}\|^2\delta < 1.$$

Hence T is indeed invertible and the hyperbolicity follows.

Next we show that $z_k \to x_0$ as $|k| \to \infty$. To this end, we prove that $\{x_0\}$ is the maximal compact invariant set inside the open ball with centre x_0 and radius

$$\varepsilon_0 = \frac{2}{M\|L^{-1}\|}.$$

We define the bounded linear operator $\tilde{L}\colon \ell^\infty(\mathbb{Z}, \mathbb{R}^n) \to \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$ by

$$(\tilde{L}\mathbf{u})_k = u_{k+1} - Df(x_0)u_k \quad \text{for } k \in \mathbb{Z}.$$

By hypothesis, L is invertible. Now we show that $\tilde{L}$ is invertible and that

$$\|\tilde{L}^{-1}\| \leq \|L^{-1}\|.$$

Let $\mathbf{g} = \{g_k\}_{k=-\infty}^{+\infty} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$. For r a natural number, the difference equation

$$v_{k+1} = Df(y_k)v_k + g_{k-r},\ k \in \mathbb{Z}$$

has a solution $v_k^{(r)}$ such that for all k

$$\|v_k^{(r)}\| \leq \|L^{-1}\|\|\mathbf{g}\|.$$

Take $u_k^{(r)} = v_{k+r}^{(r)}$. Then $u_k^{(r)}$ is a solution of

$$u_{k+1}^{(r)} = Df(y_{k+r})u_k^{(r)} + g_k \tag{18}$$

such that

$$\|u_k^{(r)}\| \leq \|L^{-1}\|\|\mathbf{g}\| \quad \text{for} \quad k \in \mathbb{Z}. \tag{19}$$

Next, by Cantor's diagonalisation procedure, we can find a subsequence $u_k^{(j_r)} \to \overline{u}_k$ as $r \to \infty$ for each k. Then letting $r \to \infty$ in (18) and (19) (with r replaced by j_r), we obtain

$$\overline{u}_{k+1} = Df(x_0)\overline{u}_k + \mathbf{g}_k$$

and

$$\|\overline{u}_k\| \leq \|L^{-1}\|\|\mathbf{g}\| \tag{20}$$

for $k \in \mathbb{Z}$. So for arbitrary $\mathbf{g}$, the difference equation

$$u_{k+1} = Df(x_0)u_k + g_k \tag{21}$$

has a solution bounded on $(-\infty, \infty)$. Thus $\tilde{L}$ is onto.

To show that $\tilde{L}$ is one to one, let P be a projection onto the sum of the generalized eigenspaces corresponding to the eigenvalues of $A = Df(x_0)$ inside the unit circle with kernel the sum of the generalized eigenspaces corresponding to the eigenvalues outside the unit circle. Then there exist positive constants K and $\lambda < 1$ such that for $k \geq 0$

$$\|A^k P\| \leq K\lambda^k, \quad \|A^{-k}(I-P)\| \leq K\lambda^k. \tag{22}$$

The difference u_k between two bounded solutions of (21) would be a bounded sequence satisfying

$$u_k = A^k u_0 = A^k P u_0 + A^k (I-P) u_0.$$

If $Pu_0 \neq 0$, then $\|A^k P u_0\| \to 0$ as $t \to \infty$ and $\|A^k P u_0\| \to \infty$ as $t \to -\infty$; on the other hand, if $(I-P)u_0 \neq \mathbf{0}$, then $\|A^k(I-P)u_0\| \to 0$ as $t \to -\infty$ and $\|A^k(I-P)u_0\| \to \infty$ as $t \to \infty$. It follows that $u_0 = 0$. Hence $\tilde{L}$ is one to one.

Thus $\tilde{L}$ is invertible so that the equation

$$u_{k+1} = Df(x_0)u_k + g_k,\ k \in \mathbb{Z}$$

has the unique bounded solution $(\tilde{L}^{-1}\mathbf{g})_k$. Thus $\overline{u}_k = (\tilde{L}^{-1}\mathbf{g})_k$. Then it follows from (20) that

$$\|\tilde{L}^{-1}\| \leq \|L^{-1}\|.$$

Now let w_k be an orbit of f such that $\sup_{k\in\mathbb{Z}} \|w_k - x_0\| < \varepsilon_0$. Then if we write $u_k = w_k - x_0$, we see that

$$u_{k+1} = Df(x_0)u_k + g_k(u_k),$$

where

$$g_k(u) = f(u + x_0) - f(x_0) - Df(x_0)u.$$

Note that

$$\|g_k(u)\| \le \frac{1}{2}M\|u\|^2.$$

Thus

$$\tilde{L}\mathbf{u} = g(\mathbf{u}),$$

where g is defined for $\mathbf{u} = \{u_k\}_{k=-\infty}^{+\infty}$ by

$$[g(\mathbf{u})]_k = g_k(u_k).$$

Then the equation can be written as

$$\mathbf{u} = \tilde{L}^{-1}g(\mathbf{u})$$

and hence

$$\|\mathbf{u}\| \le \|\tilde{L}^{-1}\|\frac{1}{2}M\|\mathbf{u}\|^2.$$

Therefore if $\|\mathbf{u}\| \neq 0$

$$1 \le \|\tilde{L}^{-1}\|\frac{1}{2}M\|\mathbf{u}\| < \|L^{-1}\|\frac{1}{2}M\varepsilon_0.$$

From the definition of ε_0, the latter is not possible and so $\|\mathbf{u}\| = 0$. It follows that $\{x_0\}$ is the maximal compact invariant set inside the open ball with centre x_0 and radius ε_0.

Now to prove that $\|z_k - x_0\| \to 0$ as $|k| \to \infty$, consider the $\omega-$limit set Ω of z_0. Since $\|z_k - x_0\| \le \varepsilon < \varepsilon_0$ for all k, it follows that Ω is contained in the open ball of radius ε_0 with centre x_0. So we deduce that $\Omega = \{x_0\}$. It follows that $\|z_k - x_0\| \to 0$ as $k \to \infty$. Similarly, we can prove that $\|z_k - x_0\| \to 0$ as $k \to -\infty$.

The proof of Theorem 6 is completed by observing that the condition in (ii) ensures that the orbit $\{z_k\}_{k=-\infty}^{+\infty}$ is distinct from the fixed point.

There are two main issues involved in applying Theorem 6. First we must find a δ pseudo homoclinic orbit with a suitably small δ. Second we must verify the invertibility of the operator L and find an upper bound for the norm of its inverse.

Finding Pseudo Homoclinic Orbits: A Global Newton's Method

First we describe how pseudo homoclinic orbits can be found. Let x_0 be a hyperbolic fixed point of f. Choose a point $\bar{y}_0$ near x_0 and find a positive integer J such that $\bar{y}_J$ is "fairly" close to x_0, where $\bar{y}_k = f^k(\bar{y}_0)$ as calculated by the computer. Form the crude (that is, large δ) finite pseudo orbit

$$x_0, \dots, x_0, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_J, x_0, \dots, x_0$$

by adding a suitable number of x_0's to both ends. Denote this extended finite pseudo orbit by $\{\bar{y}_k\}_{k=p}^q$.

We want to replace this pseudo orbit by a nearby one with a smaller δ and with the same endpoints. Ideally, we would like a sequence $\{y_k\}_{k=p}^q$ such that y_k is near $\bar{y}_k$ with

$$y_{k+1} = f(y_k), \quad k = p, \dots, q-1 \tag{23}$$

and

$$y_p = x_0, \quad y_q = x_0. \tag{24}$$

We write

$$y_k = \bar{y}_k + S_k u_k, \tag{25}$$

where $\{S_k\}_{k=p}^q$ is a sequence of orthogonal matrices defined recursively by the Gram-Schmidt process:

$$Df(\bar{y}_k)S_k = S_{k+1}A_k, \quad k = p, \dots, q-1,$$

where A_k is upper triangular. Then if we make the substitution (25) in (23), we obtain for $k = p, \dots, q-1$:

$$\bar{y}_{k+1} + S_{k+1}u_{k+1} = f(\bar{y}_k + S_k u_k).$$

Linearizing as in Newton's method, we obtain the approximate equation:

$$\bar{y}_{k+1} + S_{k+1}u_{k+1} = f(\bar{y}_k) + Df(\bar{y}_k)S_k u_k$$

and so

$$u_{k+1} = A_k u_k + g_k, \tag{26}$$

where

$$g_k = S_{k+1}^*[f(\bar{y}_k) - \bar{y}_{k+1}].$$

Notice that

$$\|g_k\| \le \delta, \quad k = p, \dots, q-1,$$

since $\{\bar{y}_k\}_{k=p}^q$ is a δ pseudo orbit.

For ease of exposition, we now restrict to $n = 2$. So let

$$A_k = \begin{bmatrix} a_k & b_k \\ 0 & c_k \end{bmatrix}$$

Then, in components, (26) takes the form

$$u^{(1)}_{k+1} = a_k u^{(1)}_k + b_k u^{(2)}_k + g^{(1)}_k, \quad u^{(2)}_{k+1} = c_k u^{(2)}_k + g^{(2)}_k.$$

Enforcing (24) would require too many conditions. The best we can do is demand that

$$u^{(2)}_p = 0,\ u^{(1)}_q = 0.$$

Note that because of the hyperbolicity, we expect that $|a_k| > 1$, $|c_k| < 1$ for most k. Then we solve

$$u^{(2)}_{k+1} = c_k u^{(2)}_k + g^{(2)}_k$$

forwards starting with $u^{(2)}_p = 0$. Next we substitute $u^{(2)}_k$ into

$$u^{(1)}_{k+1} = a_k u^{(1)}_k + b_k u^{(2)}_k + g^{(1)}_k$$

and solve backwards starting with $u^{(1)}_q = 0$.

The new pseudo orbit is $\{\bar{y}_k + S_k u_k\}_{k=p}^{q}$. The procedure is repeated until it converges. The final δ should be small enough to apply the theorem.

Invertibility of L.

To verify the invertibility of L and find a computable upper bound for $\|L^{-1}\|$, we first triangularize the matrices $Y_k = Df(y_k)$ (note for ease of exposition we ignore the roundoff error arising from the calculation of $Df(y_k)$), where we know that $Y_k = Df(x_0)$ for $k \le p$ and $k \ge q$. First we find an orthogonal matrix T and an upper triangular matrix B such that

$$Df(x_0)T = TB.$$

We can assume that the diagonal entries of B are in order of decreasing modulus. Then we take

$$S_k = T \quad \text{for} \quad k \le p+1, \quad A_k = B \quad \text{for} \quad k \le p$$

and we apply the Gram-Schmidt procedure to get orthogonal matrices S_k and upper triangular matrices A_k such that

$$Y_k S_k = S_{k+1} A_k$$

starting with $k = p+1$. We repeat this for higher values of k until we find an integer $\ell \ge q$ such that S_ℓ has, up to sign, the same columns as T (for an explanation of why this *usually* works, see Palmer [2000, p.254]). Then we adjust the signs of the entries in $A_{\ell-1}$ so that

$$Y_{\ell-1}S_{\ell-1} = TA_{\ell-1}$$

holds. Thus we may assume $S_\ell = T$. Finally, if we take

$$S_k = T \quad \text{and} \quad A_k = B \quad \text{for} \quad k \geq \ell,$$

we see that for all $k \in \mathbb{Z}$

$$Y_k S_k = S_{k+1} A_k$$

and that A_k is upper triangular with

$$A_k = B \quad \text{for} \quad k \geq \ell, \quad A_k = B \quad \text{for} \quad k \leq p.$$

To show L is invertible, given $\mathbf{g} = \{g_k\}_{k=-\infty}^{+\infty} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^n)$, we need to show that

$$v_{k+1} = Y_k v_k + g_k, \quad k \in \mathbb{Z} \tag{27}$$

has a unique bounded solution. We make the transformation

$$v_k = S_k u_k.$$

Then the equation becomes

$$u_{k+1} = A_k u_k + \overline{g}_k, \quad k \in \mathbb{Z}, \tag{28}$$

where

$$\overline{g}_k = S_{k+1}^* g_k, \quad k \in \mathbb{Z}.$$

For ease of exposition, we now restrict to $n = 2$. So let

$$A_k = \begin{bmatrix} a_k & b_k \\ 0 & c_k \end{bmatrix}$$

Then, in components, (28) reads

$$u_{k+1}^{(1)} = a_k u_k^{(1)} + b_k u_k^{(2)} + \overline{g}_k^{(1)}, \quad u_{k+1}^{(2)} = c_k u_k^{(2)} + \overline{g}_k^{(2)}, \quad k \in \mathbb{Z}. \tag{29}$$

To solve (29), we need a couple of lemmas.

Lemma 2. *Let* $\{c_k\}_{k=-\infty}^{\infty}$ *be a sequence of nonzero scalars such that*

$$c_k = c \quad \text{for} \quad k \leq p \quad \text{and} \quad k \geq \ell,$$

where $p < \ell$ *and* $|c| < 1$*. Then if* $\{h_k\}_{k=-\infty}^{\infty}$ *is a bounded sequence of scalars, the difference equation*

$$u_{k+1} = c_k u_k + h_k, \quad k \in \mathbb{Z} \tag{30}$$

has a unique bounded solution u_k. *Moreover,*

$$|u_k| \leq L \sup_{j\in\mathbb{Z}} |h_j|, \quad k \in \mathbb{Z}, \tag{31}$$

where

$$L = v_{p+1} + \max_{k=p+2}^{\ell} v_k$$

and $\{v_k\}_{k=p+1}^{\ell}$ *is defined recursively by*

$$v_{p+1} = (1-|c|)^{-1} \quad \text{and} \quad v_{k+1} = |c_k| v_k + 1, \quad k = p+1, \dots, \ell - 1.$$

Proof. Consider the expression

$$\eta_k = \sum_{m=-\infty}^{k} |c_{k-1} \cdots c_m| = 1 + |c_{k-1}| + |c_{k-1}c_{k-2}| + \ \cdots, \quad k \in \mathbb{Z}.$$

If $k \leq p+1$,

$$\eta_k = \sum_{m=-\infty}^{k} |c|^{k-m} = v_{p+1}. \tag{32}$$

Next note that for all k,

$$\eta_{k+1} = |c_k| \eta_k + 1. \tag{33}$$

So, since also $\eta_{p+1} \leq v_{p+1}$, we have

$$\eta_k \leq v_k, \quad p+2 \leq k \leq \ell. \tag{34}$$

Next by repeated application of (33), we see that for $k \geq \ell + 1$

$$\eta_k = |c_{k-1} \cdots c_\ell| \eta_\ell + \sum_{m=\ell+1}^{k} |c_{k-1} \cdots c_m| \leq \eta_\ell + \sum_{m=\ell+1}^{k} |c_{k-1} \cdots c_m|. \tag{35}$$

By a similar argument to that used above,

$$\sum_{m=\ell+1}^{k} |c_{k-1} \cdots c_m| \leq v_{p+1}, \quad k \geq \ell + 1.$$

Noting also that $\eta_\ell \leq v_\ell$, we conclude that

$$\eta_k \leq v_{p+1} + v_\ell, \quad k \geq \ell + 1. \tag{36}$$

Then in view of (32), (34), and (36) we see that we have proved

$$\eta_k \leq L \quad \text{for} \quad k \in \mathbb{Z}.$$

Then

$$u_k = \sum_{m=-\infty}^{k} c_{k-1} \cdots c_m h_{m-1}$$

is a bounded solution of (30) satisfying (31). Moreover, the difference u_k between any two bounded solutions of (30) would be a bounded solution of

$$u_{k+1} = c_k u_k \quad k \in \mathbb{Z}.$$

Then $u_{k+1} = cu_k$ for all $k \le p$ and so

$$\sup_{k \le p} |u_k| \le |c| \sup_{k \le p} |u_k|,$$

which implies $u_k = 0$ for all k. So the bounded solution u_k of (30) is unique.

Lemma 3. *Let $\{a_k\}_{k=-\infty}^{\infty}$ be a sequence of nonzero scalars such that*

$$a_k = a \quad \text{for} \quad k \le p \quad \text{and} \quad k \ge \ell,$$

where $p < \ell$ and $|a| > 1$. Then if $\{h_k\}_{k=-\infty}^{\infty}$ is a bounded sequence of scalars, the difference equation

$$u_{k+1} = a_k u_k + h_k, \quad k \in \mathbb{Z} \tag{37}$$

has a unique bounded solution u_k. Moreover,

$$|u_k| \le L \sup_{j \in \mathbb{Z}} |h_j|, \tag{38}$$

where

$$L = v_\ell + \max_{k=p+1}^{\ell-1} v_k$$

and v_k is defined backwards recursively by

$$v_\ell = |a|(|a|-1)^{-1} \quad \text{and} \quad v_k = |a_k^{-1}| v_{k+1} + |a_k^{-1}|, \quad k = \ell-1, \ldots, p+1.$$

Proof. Consider the expression

$$\eta_k = \sum_{m=k}^{\infty} |a_k^{-1} \cdots a_m^{-1}|.$$

If $k \ge \ell$,

$$\eta_k = \sum_{m=k}^{\infty} \left(\frac{1}{|a|}\right)^{m-k+1} = v_\ell. \tag{39}$$

Next note that for all k,

$$\eta_k = |a_k|^{-1} \eta_{k+1} + |a_k|^{-1}. \tag{40}$$

So, since also $\eta_\ell \le v_\ell$,

$$\eta_k \le v_k, \quad p+1 \le k \le \ell - 1. \tag{41}$$

Next by repeated application of (40), we see that for $k \leq p$

$$\eta_k = |a_k^{-1} \cdots a_p^{-1}|\eta_{p+1} + \sum_{m=k}^{p} |a_k^{-1} \cdots a_m^{-1}| \leq \eta_{p+1} + \sum_{m=k}^{p} |a_k^{-1} \cdots a_m^{-1}|.$$

By a similar argument to that used above,

$$\sum_{m=k}^{p} |a_k^{-1} \cdots a_m^{-1}| \leq v_\ell, \quad k \leq p.$$

Since also $\eta_{p+1} \leq v_{p+1}$, it follows that for $k \leq p$

$$\eta_k \leq v_{p+1} + v_\ell. \tag{42}$$

Then in view of (39), (41), and (42) we see that we have proved

$$\eta_k \leq L \quad \text{for} \quad k \in \mathbb{Z}.$$

Then we see that

$$u_k = \sum_{m=-\infty}^{k} c_{k-1} \cdots c_m h_{m-1}$$

is a bounded solution of (37) satisfying (38). Moreover, the difference u_k between any two bounded solutions of (37) would be a bounded solution of

$$u_{k+1} = a_k u_k, \quad k \in \mathbb{Z}.$$

Then $u_{k+1} = a u_k$ for all $k \geq \ell$ and so $|u_k| \to \infty$ as $k \to \infty$ unless $u_k = 0$ for all k. So the bounded solution u_k of (37) is unique and Lemma 3 is proved.

We use these two lemmas to solve (29). Note if

$$B = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

then we know that $A_k = B$ for $k \leq p$ and $k \geq \ell$. Also, since x_0 is hyperbolic, we know that $|a| > 1$ and $|c| < 1$. Therefore we may apply Lemma 2 to deduce that

$$u_{k+1}^{(2)} = c_k u_k^{(2)} + \overline{g}_k^{(2)}, \quad k \in \mathbb{Z}$$

has a unique bounded solution $u_k^{(2)}$ which satisfies

$$\|u_k^{(2)}\| \leq L_2 \|\overline{\mathbf{g}}\|$$

for all k, where

$$L_2 = v_{p+1} + \max_{k=p+2}^{\ell} v_k$$

and $\{v_k\}_{k=p+1}^{\ell}$ is defined recursively by

$$v_{p+1} = (1 - |c|)^{-1} \quad \text{and} \quad v_{k+1} = |c_k| v_k + 1, \quad k = p+1, \ldots, \ell - 1.$$

We substitute the solution $u_k^{(2)}$ in the first equation in (29) to obtain

$$u_{k+1}^{(1)} = a_k u_k^{(1)} + h_k, \quad k \in \mathbb{Z} \quad \text{where} \quad h_k = b_k u_k^{(2)} + \overline{g}_k^{(1)}.$$

By Lemma 3, this equation has a unique bounded solution $u_k^{(1)}$ which satisfies

$$\|u_k^{(1)}\| \leq L_1 \sup_{j \in \mathbb{Z}} |h_j|,$$

where

$$L_1 = v_\ell + \max_{k=p+1}^{\ell-1} v_k$$

and now v_k is defined backwards recursively by

$$v_\ell = |a|(|a|-1)^{-1} \quad \text{and} \quad v_k = |a_k^{-1}|v_{k+1} + |a_k^{-1}|, \quad k = \ell-1, \ldots, p+1.$$

Note that for all k

$$|h_k| \leq bL_2\|\overline{\mathbf{g}}\| + \|\overline{\mathbf{g}}\|,$$

where

$$b = \sup_{k \in \mathbb{Z}} |b_k|.$$

So if we use the maximum norm in $\mathbb{R}^2$, we deduce that (28) has a unique bounded solution u_k which satisfies

$$\|u_k\| \leq \max\{L_2, L_1(1+bL_2)\}\|\overline{\mathbf{g}}\|, \; k \in \mathbb{Z}.$$

Then it follows that $v_k = S_k u_k$ is the unique bounded solution of (27) and it satisfies

$$\|v_k\| \leq \max\{L_2, L_1(1+bL_2)\}\|\mathbf{g}\|, \; k \in \mathbb{Z}.$$

Hence, provided $|a| > 1$ and $|c| < 1$ (and provided we can triangularize Y_k as at the beginning), the operator L is invertible and

$$\|L^{-1}\| \leq \max\{L_2, L_1(1+bL_2)\}.$$

Example 3. Consider the Hénon map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (1 - ax^2 + y, bx)$$

with $a = 1.4$, $b = 0.3$. First we find the hyperbolic fixed point

$$x_0 = (y_1, y_2) = (0.631354, 0.189406)$$

We calculate the derivative

$$Df(x_0) = \begin{bmatrix} -2ay_1 & 1 \\ b & 0 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = 0.155946, \quad \lambda_2 = -1.923738.$$

Next, we take the point

$$\bar{y}_0 = (0.622,\ 0.188)$$

near the fixed point and observe that, as calculated by the machine,

$$\|f^{18}(\bar{y}_0) - x_0\| \leq 0.05.$$

Using the global Newton's method, we obtain a refined δ pseudo homoclinic orbit $\{y_k\}_{k=-\infty}^{+\infty}$ connecting x_0 to itself with

$$\delta = 1.95 \times 10^{-15}.$$

Next we verify the invertibility of L and find that

$$\|L^{-1}\| \leq 16.52646.$$

Set

$$M = \sup_{x \in \mathbb{R}^2} \|D^2 f(x)\| = 2.8 \quad \text{and} \quad \varepsilon = 2\|L^{-1}\|\delta \leq 3.23 \times 10^{-14}.$$

Then we verify the inequality

$$2M\|L^{-1}\|^2\delta < 1$$

and notice that one point $(-0.939, 0.358)$ on the pseudo homoclinic orbit is at a distance exceeding ε from the fixed point x_0.

Thus we conclude that the pseudo homoclinic orbit $\{y_k\}_{k=-\infty}^{+\infty}$ is ε-shadowed by a unique true orbit $\{z_k\}_{k=-\infty}^{+\infty}$ such that z_0 is a transversal homoclinic point to the fixed point x_0.

4. Other Aspects of Numerical Shadowing

We can also consider homoclinic orbits to periodic orbits. These may have a *phase shift.* When there is no phase shift, they correspond to homoclinic orbits to a fixed point but when there is a phase shift they correspond to heteroclinic orbits. Note however from the numerical point of view it is preferable to work with the original map and not an iterate, especially when the period is high. Also in this case we need to find the periodic orbits by shadowing also, especially when the period is high.

Example 4. Hénon studied the *area-preserving* quadratic map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \cos\alpha - (x_2 - x_1^2)\sin\alpha \\ x_1 \sin\alpha + (x_2 - x_1^2)\cos\alpha \end{pmatrix}$$

depending on a parameter α. When $\alpha = 1.32843$, the origin is a stable elliptic fixed point. Also, there is a hyperbolic periodic orbit of period five. In the vicinity of this periodic orbit, the dynamics appear to be chaotic.

We construct a δ pseudo homoclinic orbit, with $\delta = 2.17 \times 10^{-15}$ to the period five orbit. Using a more general Homoclinic Shadowing Theorem, we can prove the existence of a true transversal homoclinic orbit within $\varepsilon = 4.74 \times 10^{-10}$ of this pseudo orbit. This homoclinic orbit has a *phase shift* of 1.

This kind of theory can also be carried out for *autonomous systems of ordinary differential equations*. We first consider hyperbolic periodic orbits and give conditions under which a numerically computed apparent periodic orbit can be verified to be shadowed by a true hyperbolic periodic orbit. Then we consider orbits which are close to being homoclinic to this periodic orbit and give conditions under which there is a true transversal homoclinic orbit nearby.

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Note that Palmer [2000] contains a fairly comprehensive bibliography of shadowing up to 2000. More recent references can be found in the bibliographies of the other papers.

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