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ON THE NUMBER OF GENERATORS OF THE HOMOGENEOUS IDEALS OF PROJECTIVE CURVES

YOUNGOOK CHOI

Let X be a nondegenerate n-dimensional projective curve in $\mathbb{P}^r$, defined over a complex number field $\mathbb{C}$. A projective variety $X \subset \mathbb{P}^r$ is called a quasi-complete intersection if X is locally Cohen-Macaulay and cut out scheme-theoretically by $(r-n+1)$ hypersurfaces in $\mathbb{P}^r$. If X is a codimension 2 subvariety in $\mathbb{P}^r$, X is a quasi-complete intersection if the sheaf of ideal $\mathcal{I}_X$ has a following exact sequence;

$$0 \to \mathcal{E} \to \oplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^r}(-d_i) \xrightarrow{\varphi} \mathcal{I}_X \to 0, \tag{0.1}$$

where $\mathcal{E}$ is a kernel of φ and d_1, d_2, d_3 are the degrees of the hypersurfaces defining X. Note that $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^r$ because X is locally Cohen-Macaulay. A variety X is called a quasi-complete intersection curve of type (d_1, d_2, d_3) if X is defined scheme-theoretically by three surfaces f_1, f_2, f_3 of degrees d_1, d_2, d_3 in I_X with $d_1 \geq d_2 \geq d_3$. In this talk, we are mainly interested in a projective curve X in $\mathbb{P}^3$. We discuss minimal free resolutions of the homogeneous ideals of quasi-complete intersection space curves.

Theorem 0.1. [2] *Let $X \subset \mathbb{P}^3$ be a quasi-complete intersection space curve of type (d_1, d_2, d_3), then I_X has a following minimal free resolution;*

$$0 \to \oplus_{i=1}^{\mu-3} S(d_{i+3} + c_1) \to \oplus_{i=1}^{2\mu-4} S(-e_i) \to \oplus_{i=1}^{\mu} S(-d_i) \to I_X \to 0,$$

where $d_i, e_i \in \mathbb{Z}$ and $c_1 = -d_1 - d_2 - d_3$.

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Therefore the rank of the first and the second syzygy modules are determined by the number of elements in a minimal generating set of I_X. To prove Theorem 0.1, we use the following results of the first cohomology group of a rank two vector bundle on $\mathbb{P}^3$. Let $\mathcal{E}$ be a rank two vector bundle on $\mathbb{P}^3$ and let $M_{\mathcal{E}} = H^1_*(\mathcal{E})$. Assume that $M_{\mathcal{E}}$ has a following minimal free resolution over $S = K[x_0, x_1, x_2, x_3]$:

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \to L_0 \to L_{\mathcal{E}} \to 0,$$

where rank $L_0 = s$ and rank $L_1 = t$. Decker and Rao ([3], Proposition 1 and [6], Corollary 2.3) independently show that $t = 2s + 2$, i.e., the rank of the first syzygy module is determined by the number of minimal generators of $M_{\mathcal{E}}$.

One corollary of this result is that for a given positive integer $t \in \mathbb{Z}^+$, one can construct a smooth quasi-complete intersection curve X such that the number of minimal generators of I_X is t. Let X be a smooth elliptic curve in $\mathbb{P}^3$ of degree $t+4$. Since $\omega_X = \mathcal{O}_X$, X is the zero locus of a section of a rank two vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ with $c_1(\mathcal{F}) = 4$ and $c_2(\mathcal{F}) = t + 4$ ([6], [5]). If $\mathcal{E} = \mathcal{F}(-2)$, we have;

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{E}(-2) \to \mathcal{I}_X \to 0.$$

Since $H^1(\mathcal{P}^3, \mathcal{I}_X(l)) = 0$ for $l \leq 0$, $\mathcal{E}$ is a mathematical instanton bundle. Since $H^1(\mathcal{E}(-1)) = H^1(\mathcal{I}_X(1)) = t$, $M_{\mathcal{E}}$ has a following minimal free resolution;

$$\oplus S(-d_i) \to S^{\oplus 2t+2} \to S(1)^{\oplus t} \to M_{\mathcal{E}} \to o.$$

By Theorem 2.5 in [4], a residual Y of X in a complete intersection of two surfaces of degree d_1 and d_2 is a quasi-complete intersection of type $(d_1 + d_2 - 4, d_1, d_2)$. By Theorem 0.1, the homogeneous ideal I_Y has a following minimal free resolution;

$$0 \to \oplus_{i=1}^t S(f_i + c_1) \to L_1 \to \oplus_{i=1}^t S(-f_i) \oplus S(-d_1) \oplus S(-d_2) \oplus S(-d_1 - d_2 + 4) \to I_Y \to 0$$

where $c_1 = -(2d_1 + 2d_2 - 4)$ and $f_i \in \mathbb{Z}$. Therefore, for any given number t, one can construct a quasi-complete intersection curve Y in $\mathbb{P}^3$ such that the number of minimal generators of I_Y is t.

Another corollary is that one can give a characterization of quasi-complete intersections whose ideals are generated by four homogeneous polynomials. We show that the ideal of X is generated by four elements if and only if the Hartshorne-Rao module M_X is a "complete intersection" graded S-module with one generator, i.e. $M \simeq S(-d)/(f_1, f_2, f_3, f_4)$ where f_1, f_2, f_3, f_4 is a regular sequence and $d \in \mathbb{Z}$. By the way, a quasi-complete intersection whose ideal is generated by two elements is a complete intersection and a quasi-complete intersection whose ideal is generated by three elements is arithmetically Cohen-Macaulay. The geometry of these curves is well-known.

Finally, we note that a minimal free resolution of the homogeneous ideal of a quasi-complete intersection curve is similar as that of a monomial curve. A curve $X \subset \mathbb{P}^3$ is called a monomial curve if it has a parametric representation $(t_0^{n_3}, t_0^{n_3-n_1}t_1^{n_1}, t_0^{n_3-n_2}t_1^{n_2}, t_1^{n_3})$ where $n_1 < n_2 < n_3 \in \mathbb{Z}$ with g.c.d.$(n_1, n_2, n_3) = 1$. Let $I_X \subset S = K[x_0, x_1, x_2, x_3]$ be the homogeneous ideal of a monomial curve in $\mathbb{P}^3$ and let $\mu(I_X)$ be the number of elements in a minimal generating set of I_X. In his paper ([1]), Bresinsky proves that if $\mu(I_X) = \mu \geq 3$, then I_X has a following minimal free resolution;

$$0 \to \sum_{j=1}^{\mu-3} S(-d_{3,j}) \to \sum_{j=1}^{2\mu-4} S(-d_{2,j}) \to \sum_{j=1}^{\mu} S(-d_{1,j}) \to I_X \to 0,$$

where $d_{i,j} \in \mathbb{Z}$.

References

[1] H. Bresinsky, Minimal free resolutions of monomial curves in $\mathbb{P}^3$, *Linear alg. and its appl.* **59** (1984) 121-129.

[2] Y. Choi, On the syzygies of quasi-complete intersection space curves, submitted.

[3] W. Decker, Monads and cohomology modules of rank 2 vecotr bunldes, *Compositio Math.* **76** (1990) 7–17

[4] D. Franco, L. Kleiman, T. Lascu, Gherardelli linkage and complete intersections, *Michigan Math. J.* **48** (2000) 271–279.

[5] R. Hartshorne, Stable vector bundles of rank 2 on $\mathbb{P}^3$, *Math. Ann.* **238** (1978) 229–280.

[6] P. Rao, A note on cohomology modules of rank two bundles, *J. Algebra* **86** (1984) 23–34.