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# WEAK AND STRONG CONVERGENCE OF THE ISHIKAWA ITERATION PROCESS WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we prove the weak and strong convergence of the Ishikawa iterative scheme with errors to a common fixed point for two asymptotically nonexpansive mappings in a uniformly convex Banach space under a condition weaker than compactness. Our theorems improve and generalize recent known results in literature.

## 1. Introduction

Let $K$ be a nonempty subset of a real normed linear space $E$. Let $T$ be a self-mappings of $K$. $T$ is said to be asymptotically nonexpansive with constant $\mu_n$ if there exists $\mu_n \in [0, +\infty)$, $\lim_{n\to\infty} \mu_n = 0$, such that

$$\|T^n(x) - T^n(y)\| \le (1 + \mu_n)\|x - y\|, \quad \forall x, y \in K.$$

$T$ is called nonexpansive if $\|T(x) - T(y)\| \le \|x - y\|, \quad \forall x, y \in K$.

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, but the converse does not hold.

It was proved in [1] that if $E$ is uniformly convex and if $K$ is bounded, closed, and convex, then every asymptotically nonexpansive mapping has a fixed point.

Takahashi and Tamuro([6]) introduced the following iterative schemes known as Ishikawa iterative schemes for a pair of nonexpansive mappings;

$$(1.1) \qquad \begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n S y_n + (1 - a_n) x_n, \\ y_n = b_n T x_n + (1 - b_n) x_n, \quad n \ge 1, \end{cases}$$

where $a_n, b_n \in [0, 1]$.

Khan and Hafiz([3]) generalized the scheme (1.1) to the one with errors for a pair of nonexpansive mappings as follows;

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$$(1.2)\qquad \begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n S y_n + b_n x_n + c_n u_n, \\ y_n = a_n^{'} T x_n + b_n^{'} + c_n^{'} v_n, \quad n \geq 1, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, $\{c_n^{'}\}$ are sequences in $[0,1]$ with $0 < \delta \leq a_n, a_n^{'} \leq 1-\delta < 1$, $a_n + b_n + c_n = 1 = a_n^{'} + b_n^{'} + c_n^{'}$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in $K$.

We further generalize this scheme (1.2) for a pair of asymptotically nonexpansive mappings as follows;

$$(1.3)\qquad \begin{cases} x_1 = x \in K, \\ x_{n+1} = a_n S^n y_n + b_n x_n + c_n u_n, \\ y_n = \bar{a}_n T^n x_n + \bar{b}_n x_n + \bar{c}_n v_n, \quad n \geq 1, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0,1]$ with $0 < \delta \leq a_n, \bar{a}_n \leq 1-\delta < 1$, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in $K$.

In this paper, we study the Ishikawa iterative scheme with error numbers (1.3) for the weak and strong convergence for a pair of asymptotically nonexpansive mappings in a uniformly convex Banach space. Our theorems improve and generalize some previous results.

## 2. Preliminaries

Let $E$ be a Banach space and let $K$ be a nonempty subset of $E$. Let $T$ be a mapping of $K$ into itself. For every $\varepsilon$ with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of $E$ by

$$\delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\}.$$

A Banach space $E$ is said to be uniformly convex if $\delta(\varepsilon) > 0$. A uniformly convex Banach space is reflexive and strictly convex.

A Banach space $E$ is said to satisfy Opial's condition([4]) if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

We first prove the following lemma.

**Lemma 2.1.** Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\mu_n\}$ be four nonnegative sequences satisfying

$$a_{n+1} \leq (1+\gamma_n)(1+\mu_n)a_n + \beta_n$$

for all $n \geq 1$. If $\sum_{i=1}^{\infty} \mu_n < \infty$, $\sum_{i=1}^{\infty} \gamma_n < \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

**Proof.** By hypothesis, we obtain

$$\begin{aligned}
a_{n+m} &\le (1+\gamma_{n+m-1})(1+\mu_{n+m-1})a_{n+m-1}+\beta_{n+m-1}\\
&\le e^{\gamma_{n+m-1}+\mu_{n+m-1}}a_{n+m-1}+\beta_{n+m-1}\\
&= e^{\gamma_{n+m-1}+\mu_{n+m-1}}[(1+\gamma_{n+m-2})(1+\mu_{n+m-2})a_{n+m-2}+\beta_{n+m-2}]+\beta_{n+m-1}\\
&\le e^{(\gamma_{n+m-1}+\gamma_{n+m-2})+(\mu_{n+m-1}+\mu_{n+m-2})}a_{n+m-2}\\
&\quad + e^{\gamma_{n+m-1}+\mu_{n+m-1}}\beta_{n+m-2}+\beta_{n+m-1}\\
&\cdots\\
&\le e^{\sum_{k=n}^{n+m-1}(\gamma_k+\mu_k)}a_n + e^{\sum_{k=n}^{n+m-1}(\gamma_k+\mu_k)}\sum_{k=1}^{n+m-1}\beta_k.
\end{aligned}$$

Thus, from $\sum_{n=1}^{\infty}\gamma_n < +\infty$, $\sum_{n=1}^{\infty}\mu_k < +\infty$, $\sum_{n=1}^{\infty}\beta_n < +\infty$, we can obtain $\limsup_{n\to\infty} a_n \le \liminf_{n\to\infty} a_n$. So, $\lim_{n\to\infty} a_n$ exists.

**Lemma 2.2(see [2]).** Let $E$ be a uniformly convex Banach space satisfying Opial's condition, $\phi \ne K \subset E$ closed and convex, and $T: K \to K$ asympototically nonexpansive. Then $I - T$ is demiclosed with respect to zero.

We also know the following lemma proved by Schu([5]).

**Lemma 2.3.** Let $E$ be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $E$ such that $\limsup_{n\to\infty}\|x_n\| \le r$, $\limsup_{n\to\infty}\|y_n\| \le r$, and $\limsup_{n\to\infty}\|\alpha_n x_n + (1-\alpha_n)y_n\| = r$ holds for some $r \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

## 3. Main Results

In this section, we prove our main theorems. Let $K$ be a nonempty bounded convex subset of a real uniformly convex Banach space $E$. Let $S, T: K \to K$ be asymptotically nonexpansive mappings. Let $F(S)$ denote the set of all fixed points of $S$. The following iteration scheme is studied:

$$\begin{cases} x_1 = x \in K,\\ x_{n+1} = a_n S^n y_n + b_n x_n + c_n u_n,\\ y_n = \bar{a}_n T^n x_n + \bar{b}_n x_n + \bar{c}_n v_n, \quad n \ge 1, \end{cases} \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a}_n\}$, $\{\bar{b}_n\}$, $\{\bar{c}_n\}$ are sequences in $[0,1]$ with $0 < \delta \le a_n, \bar{a}_n \le 1-\delta < 1$, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$, and $\{u_n\}$, $\{v_n\}$ are bounded sequences in $K$.

**Theorem 3.1.** Let $E$ be a uniformly convex Banach space satisfying the Opial's condition and $K, S, T$ and $\{x_n\}$ be as taken in Lemma 3.3. If $F(S) \cap F(T) \ne \phi$, then $\{x_n\}$ converges weakly to a common fixed point of $S$ and $T$.

**Theorem 3.2.** Let $E$ be a uniformly convex Banach space and $K$, $\{x_n\}$ be as taken in Lemma 3.3. Let $S, T: K \to K$ be two asymtotically nonexpansive mappings satisfying condition (A). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.

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