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UNIFORM DECAY FOR THE SOLUTION OF THE PLATE EQUATION WITH A BOUNDARY CONDITION OF MEMORY TYPE

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Abstract. In this paper, we study the stability of solutions for the plate equation with a boundary condition of memory type. We show that such dissipation is strong enough to produce exponential decay to the solution, provided the relaxation functions also decays exponentially.

1. Introduction

⊨ The main purpose of this work is to study the asymptotic behavior of the solution of the plate equation with a boundary condition of memory type. For this, let Ω be a open boundary set of R^2 with regular boundary Γ. We divide the boundary into two parts: $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ, and let us denote by $\eta = (-\nu_2, \nu_1)$ the unit tangent positively oriented on Γ. We consider the following initial boundary problem:

$$u_{tt} + \Delta^2 u + F(x, t, u, \Delta u) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

$$u = \frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \tag{1.2}$$

$$-u + \int_0^t g_1(t-s)\mathfrak{B}_2 u(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{1.3}$$

$$\frac{\partial u}{\partial v} + \int_0^t g_2(t-s)\mathfrak{B}_1 u(s)ds = 0, \quad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{1.4}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.5}$$

where

$$\mathfrak{B}_1 u = \Delta u + (1-\mu)B_1 u,$$
$$\mathfrak{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

and

$$B_1 u = 2\nu_1\nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2},$$
$$B_2 u = (\nu_1^2 - \nu_2^2)\frac{\partial^2 u}{\partial x \partial y} + \nu_1\nu_2(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}).$$

In (1.2), u denote the positive of the plate. The integral equations (1.4) and (1.5) describe the memory effects which can be caused, for example, by the interaction with another viscoelastic element. By g_1 and g_2 we are denote the relaxation functions. This system models the small transversal vibrations of a thin plate whose Poisson coefficient is equal to μ, with $\mu \in (0, \frac{1}{2})$(see Lagnese and Lagnese[*] or Lions[4]).

We assume that there exists $x_0 \in R^2$ such that

$$\Gamma_0 = \{x \in \Gamma : \nu(x) \cdot (x - x_0) \leq 0\}, \tag{1.6}$$

$$\Gamma_1 = \{x \in \Gamma : \nu(x) \cdot (x - x_0) > 0\}. \tag{1.7}$$

If we denote by $m(x) = x - x_0$, the compactness of Γ_1 and condition (1.8) imply that there exist a small positive constant δ_0 such that

$$0 < \delta \leq m(x) \cdot \nu(x), \quad \forall \in \Gamma_1. \tag{1.8}$$

Resently, many authors have studied the existence and the uniform decay stabilization of solutions for a variety of wave equations [*]. Resently, M.M. Cavalcanti[*] proved the general decay rates of solutions to a nonlinear wave equation with boundary condition of memory type. M.L.Santos and F.Junior[*] investigated the stability of solutions for Kirchhoff plates equation (1.1)-(1.5) with memory condition working at the boundary when $F = 0$. Moreover, J.E.Munoz Rivera, H.Portillo Oquendo, M.L.Santos[*] consider the stability of solutions to a von Karman system for Kirchhoff plate equations with a memory condition working at the boundary.

Stability of problems with the nonlinear term $F(x, t, u, \Delta u)$ require a careful treatment, because we do not have any information about the influence of integral $\int_\Omega F(x, t, u, \Delta u)u' dx$ on the equivalent energy $E(t)$ or about the sign of the derivative $E'(t)$. In other words, we cannot guarantee that $E'(t) \leq 0$, which plays an essential role in establishing the desired decay rates.

The main result this paper is to show that the solutions of system (1.1)-(1.5) decays uniformly in time with the same rate of decay of the relaxation function. Moreover precisely, denoting by k the resolvent kernel of $-\frac{g'}{g(0)}$, we show that the solution decays exponentially to zero provided k decays exponentially to zero.

The method used here is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1 \mathcal{L}(t) + c_2 e^{-\gamma t}$$

for some positive constants c_1, c_2 and γ.

Because of condition (1.2) the solution of the system (1.1)-(1.5) must belong to the following space:

$$W = \{w \in H^2(\Omega); w = \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \ \Gamma_0\}.$$

Let us define the bilinear form $a(\cdot,\cdot)$ as follows:

$$a(u,v)=\int_\Omega\Big\{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}+\mu\Big(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\Big)$$
$$+2(1-\mu)\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y}\Big\}dxdy.$$

The following Green's formula will be often used.

Lemma 1.1. For any u and v be functions in $H^4(\Omega)\cap W$. Then we have

$$\int_\Omega(\Delta^2 u)vdx=a(u,v)+\int_{\Gamma_1}\Big\{(\mathfrak{B}_2 u)v-(\mathfrak{B}_1 u)\frac{\partial v}{\partial\nu}\Big\}d\Gamma_1.$$

The notation used in this paper is standard and can be found in Lions's book[4]. In the sequel by c(sometimes $c_1,c_2,\cdots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In Section 2, we give some notations, assumptions and main result. In Section 3, we prove the existence of solutions of the problems (1.1)-(1.5) and the uniform decay of energy is given in Section 4.

2. Notaions and main results

In this section we shall study the existence and regularity of solutions to Eqs. (1.1)-(1.5). To this end we will assume that the relaxation functions g_1 and g_2 are positive and we shall use equations (1.3) and (1.4) to estimate the value $\mathfrak{B}_1$ and $\mathfrak{B}_2$ on Γ_1. Denoting by

$$(g*\varphi)(t)=\int_0^s g(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating (1.3) and (1.4) we arrive to the following Volterra equations:

$$\mathfrak{B}_2 u+\frac{1}{g_1(0)}g_1'*\mathfrak{B}_2 u=\frac{1}{g_1(0)}u',$$
$$\mathfrak{B}_1 u+\frac{1}{g_2(0)}g_2'*\mathfrak{B}_1 u=-\frac{1}{g_2(0)}\frac{\partial u'}{\partial\nu}.$$

Applying the Volterra's inverse operator, we get

$$\mathfrak{B}_2 u=\frac{1}{g_1(0)}\{u'+k_1*u'\},$$
$$\mathfrak{B}_1 u=-\frac{1}{g_2(0)}\{\frac{\partial u'}{\partial\nu}+k_2*\frac{\partial u'}{\partial\nu}\},$$

where the resolvent kernels satisfies

$$k_i+\frac{1}{g_i(0)}g_i'*k_i=\frac{1}{g_i(0)}g_i',\quad\forall i=1,2.$$

Denoting by $\tau_1=\frac{1}{g_1(0)}$ and $\tau_2=\frac{1}{g_2(0)}$ can be written as

$$\mathfrak{B}_2 u=\tau_1\{u'+k_1(0)u-k_1(t)u_0+k_1'*u\},$$
$$\mathfrak{B}_1 u=\tau_2\{-\frac{\partial u'}{\partial\nu}-k_2(0)\frac{\partial u}{\partial\nu}+k_2(t)\frac{\partial u_0}{\partial\nu}-k_2'*\frac{\partial u}{\partial\nu}\}.$$

Lemma 2.1. If h is a positive continuous function, then k also is a positive continuous function, Moreover,

1. If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that $h(t)) \leq c_0 e^{-\gamma t}$ then the function k satisfies $k(t) \leq \frac{c_0(\gamma-\epsilon)}{\gamma-\epsilon-c_0} e^{-\epsilon t}$, for all $0 < \epsilon < \gamma - c_0$.

2. Given $p > 1$, let us denote by $c_p = \sup_{t\in\mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that $h(t) \leq c_0(1+t)^{-p}$ then the function k satisfies $k(t) \leq \frac{c_0}{1-c_0 c_p}(1+t)^{-p}$.

Lemma 2.2. For $g, \varphi \in C^1([0,\infty) : R)$ we have

$$\int_0^t g(t-s)\varphi(s)ds\varphi_t = -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g' \diamond \varphi - \frac{1}{2}\frac{d}{dt}[g \diamond \varphi - (\int_0^t g(s)ds)|\varphi|^2].$$

Lemma 2.3. Suppose that $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_1)$ and $h \in H^{3/2}(\Gamma_1)$ then, any solution of

$$a(v,w) = \int_\Omega fwdx + \int_{\Gamma_1} gwd\Gamma_1 + \int_{\Gamma_1} h\frac{\partial w}{\partial v}d\Gamma_1, \quad \forall w \in W$$

satisfies

$$v \in H^4(\Omega)$$

and also

$$\begin{aligned}
&\Delta^2 = f,\\
&v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0,\\
&\mathfrak{B}_1 v = h, \quad \mathfrak{B}_2 v = g \quad \text{on} \quad \Gamma_1.
\end{aligned}$$

Lemma 2.4. Let f be a real positive function of class C^1. If there exists positive constants γ_0, γ_1 and c_0 such that

$$f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$f(t) \leq (f(0)+c)e^{-\gamma t}.$$

Now, we state the general hypotheses.

(A1) suppose $F : \Omega \times [0,\infty) \times R^2 \to R$ is an element of the space $C^1(\bar{\Omega} \times [0,\Omega) \times R \times R^n)$ and satisfies

$$F(x,t,\xi,\zeta) \leq c_0(1+|\xi|^{r+1}+|\zeta|), \tag{2.1}$$

where c_0 is a positive constant.

Let r be a positive constant for $n = 1, 2$, and $0 < r \leq 2/n - 2$ for $n \geq 3$.

Assume that there is a non-negative decreasing function $C(t)$ in the space $L^\infty(0,\infty) \bigcap L^1(0,\infty)$, such that

$$F(x,t,\xi,\zeta) \geq |\xi|^\gamma \xi\eta - C(t)(1 + |\eta||\xi|) \quad \forall \eta \in R. \tag{2.2}$$

Assume that there exist positive constants c_1, c_2 and c_3 such that

$$|F_t(x,t,\xi,\zeta)| \leq c_1(1 + |\zeta|^{\gamma+1} + |\xi|), \tag{2.3}$$

$$|F_\xi(x,t,\xi,\zeta)| \leq c_2(1 + |\xi|^\gamma), \tag{2.4}$$

$$|F_\zeta(x,t,\xi,\zeta)| \leq c_3. \tag{2.5}$$

We also assume that there exist positive constants D_1, D_2 such that for $\eta, \hat{\eta}$ in R and for all ξ, ζ in R,

$$\begin{aligned} &F(x,t,\xi,\zeta) - F(x,t,\hat{\xi},\hat{\zeta})(\eta - \hat{\eta}) \\ &\geq -D_1(|\xi|^r + |\hat{\xi}|^r)|\xi - \hat{\xi}||\eta - \hat{\eta}| - D_2|\eta - \hat{\eta}||\xi - \hat{\xi}|. \end{aligned} \tag{2.6}$$

Theorem 2.1. *Let $k_i \in C^2(R^+)$ be such that*

$$k_i, -k_i', -k_i'' \geq 0 \quad \textit{for} \quad i = 1, 2.$$

If $(u_0, u_1) \in (H^4(\Omega) \cap W) \times W$ satisfy the compatibility condition

$$\mathfrak{B}_2 u_0 - \tau_1 u_1 = 0 \quad \textit{on} \quad \Gamma_1,$$

$$\mathfrak{B}_1 u_0 + \tau_2 \frac{\partial u_1}{\partial \nu} = 0 \quad \textit{on} \quad \Gamma_1$$

then there exists only one solution (1.1)-(1.5) satisfying

$$u \in L^\infty(0,T; H^4(\Omega) \cap W), \quad u' \in L^\infty(0,T;W), \quad u'' \in L^\infty(0,T;L^2(\Omega)).$$

Theorem 2.2. Let us suppose that the initial data $(u_0, u_1) \in W \times L^2(\Omega)$ and that the resolvents k_1 and k_2 satisfies the conditions (3.1). Then there exist positive constants α_1 and γ_1 such that

$$E(t) \leq \alpha_1 e^{-\gamma_1 t} E(0)$$

for all $t \geq 0$.

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