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ENERGY DECAY FOR A LOCALIZED DISSIPATIVE WAVE EQUATION IN AN EXTERIOR DOMAIN

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Abstract. We derive a fast decay rate estimate of the local energy for the wave equation with a localized dissipation of the type $a(x)u_t$ in an exterior domain Ω. The dissipative coefficient $a(x)$ is nonnegative function only on a neighborhood of some part of the boundary $\partial\Omega$ and no growth conditions are imposed on the boundary.This extends some results of Nakao as well as the well-known most classical results. The method of proof is based on multipliers technique, on some interpolation inequality and differential inequality and on a similar idea of Zuazua and Nakao.

1. Introduction

In this paper we consider the decay property of the local energy of the solutions to the initial-boundary value problem for the wave equation with dissipation:

$$u_{tt} - \triangle u + a(x)u_t = 0 \ \text{ in } \ \Omega \times [0,\infty), \tag{1.1}$$

$$u = 0 \ \text{ on } \ \partial\Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \ \text{ in } \ \Omega, \tag{1.3}$$

where Ω is an exterior domain in R^N, $N \geq 1$ such that $V = \Omega^c (\equiv R^N \setminus \Omega)$ is a compact set in R^N, the boundary $\partial\Omega$ is smooth, $a(x)$ is a nonnegative function supported only on a neighborhood a part of the boundary $\partial\Omega$ and (u_0, u_1) belong to $H_0^1(\Omega) \times L^2(\Omega)$ and has a compact support, that is,

$$\text{supp } u_0 \cup \ \text{ supp } u_1 \subset B_L \equiv \{x \in R^N | |x| \leq L\}$$

for some $L > 0$.

As usually, we define the local energy $E_{loc}^R(t)$ and the total energy $E(t)$ of the solution u to the problem $(1.1) - (1.3)$ by

$$E_{loc}^R(t) \equiv \frac{1}{2}\int_{\Omega\cap B_R} (|u_t|^2 + |\nabla u|^2)dx \ \text{ and } \ E(t) \equiv \frac{1}{2}\int_{\Omega} (|u_t|^2 + |\nabla u|^2)dx, \tag{1.4}$$

where B_R is a ball with radius $R > L > 0$.

This problem has been studied by several authors (see [1, 2, 7, 9, 10, 11, 15]). It is well known that decay property of the local energy $E_{loc}^R(t)$ depends on the geometrical shape and dimension N of the domain Ω, or the dissipative term.

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When $a(x) \equiv 0$, Morawetz, Ralston and Strauss [7] proved that if Ω is nontrapping, then the local energy $E^R_{loc}(t)$ decays to 0 as $t \to \infty$, and furthermore, if V is star-shaped, then $E^R_{loc}(t)$ decays algebraically, in particular, the result of Morawetz [6] show that it decays exponentially if $N \geq 3$ and odd. Bloom and Kazarinoff [1] proved that this result holds true for more general classes of finite bodied including ones of dog-bone type. However, if V is trapping, then we cannot expect any uniform decay of $E^R_{loc}(t)$ (see [13]).

When $a(x) > 0$, we can also consider this problem. In the case $a(x) \equiv$ const.($>$ 0), Dan and Shibata [2], based on a spectral analysis, proved that

$$E^R_{loc}(t) = \mathcal{O}(t^{-N}) \ (N \geq 2),$$

and it seems to be difficult to apply to the case that $a(x)$ is not constant. On the other hand, where $a(x)$ vanishes somewhere in Ω the (local) energy decay is never trivial. In order to consider this problem, we first introduce a part of the boundary $\partial\Omega$ as follow (see Lions [5] and Russell [14]).

$$\Gamma(x_0) = \{x \in \partial\Omega | (x - x_0) \cdot \nu(x) > 0\},$$

where $x_0 \in R^N$ is arbitrarily fixed and $\nu(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$. Note that V is star-shaped with respect to x_0 if and only if $\Gamma(x_0) = \emptyset$.

Now let's state our precise assumptions on the dissipative term $a(x)u_t$.

Hypothesis A. Let $a(x) \in L^\infty(\Omega)$ is a nonnegative bounded function on Ω with $suppa(\cdot) \subset B_L$ for some $L > 0$ and there exist a relatively open set ω satisfying $\overline{\Gamma(x_0)} \subset \omega \subset \Omega$ and $x_0 \in R^N$ such that

$$a(x) \geq \varepsilon_0 > 0 \ \text{ a.e. in } \ \omega \tag{1.5}$$

or

$$a(x) > 0 \ \text{ a.e. in } \ \omega, \ \int_\omega \frac{1}{a(x)^p} dx < \infty \tag{1.6}$$

for some $0 < p < 1$. Here we remark that if the function $a(x)$ satisfies (1.6), then $a(x)$ might be zero at some point $x \in \Omega$ and thus the decay problem is more delicate to treat. Under this hypothesis, Nakao [9] considered decay estimates of local energy. In fact he proved that when $a(x)$ satisfies (1.5), any finite solution $u(t)$ satisfies

$$E^{L+\varepsilon t}_{loc}(t) = \mathcal{O}(t^{-(1-\delta)})$$

for any $0 < \varepsilon, \delta < 1$. Moreover let $m > 0$ be a integer, $N < 2m$, $\partial\Omega$ be of C^{m+1} class, and $(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfy the compatibility condition of the m-th order relative to the problem $(1.1) - (1.3)$(see [8]). If $a(x)$ satisfies (1.6) and belongs to $C^{m-1}(\bar{\Omega} \cap B_L)$, then the result of [9] show that the local energy of energy solution goes to zero as $t \to \infty$ algebraically, depending on m, that is,

$$E^{L+\varepsilon t}_{loc}(t) = \mathcal{O}\left(t^{-\frac{2mp}{2mp+N}}\right).$$

The main purpose of this paper is to derive fast decay estimates for $E^R_{loc}(t)$ without any geometric condition on $\partial\Omega$ when $a(x)$ is effective only on a part of $\partial\Omega$. We note here that our result extends Theorem 2 of Nakao [9]. For the proof of our result we provide a direct method, base on multiplier techniques combined with some ideas in Lions [5], Nakao [9] and Zuazau [15]. For types of total energy decay property see [10] and [11].

2. Preliminaries and Statement of the Main Result

Throughout this paper we shall use familiar Sobolev spaces $H^m(\Omega)$ with respect to the norm

$$\|f\|_{H^m(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha f\|_{L^2(\Omega)},$$

where $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}, ..., \partial x_N^{\alpha_N}), \alpha = (\alpha_1, ..., \alpha_N), \alpha_i \ge 0, i = 1, ..., N, |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, and $\|\cdot\|_{L^p(\Omega)}$ denotes L^P norm on Ω. The space $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. We begin with the following well-known lemmas, which we will use in the proof of the main result.

Lemma 2.1. (*Gagliardo-Nirenberg inequality*) *Let* $1 \le r < p \le \infty, 1 \le q \le p$, *and* $m \ge 0$. *Then we have the inequality*

$$\|v\|_{W^{k,p}(\Omega)} \le C\|v\|^\theta_{W^{m,q}(\Omega)} \|v\|^{1-\theta}_{L^r(\Omega)} \quad for \quad v \in W^{m,q}(\Omega) \cap L^r(\Omega)$$

with some $C > 0$ *and*

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \theta \le 1$ ($0 < \theta < 1$ *if* $p = \infty, mq =$ *integer*).

Lemma 2.2. (*General Hölder inequality*) *Let* $1 \le p_1, ..., p_m \le \infty$ *with* $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1$ *and assume* $u_k \in L^{p_k}(\Omega)$ *for* $k = 1, 2, ..., m$. *Then we have the inequality*

$$\int_\Omega |u_1 \cdots u_m| dx \le \prod_{k=1}^m \|u_i\|_{L^{p_k}(\Omega)}.$$

Let Ω be an exterior domain in R^N, $N \ge 1$ with the smooth boundary and $a(\cdot) : \overline{\Omega} \to [0, \infty)$ a nonnegative function satisfying Hypothesis. The existence and the regularity of the solution u of the problem $(1.1) - (1.3)$ is given by the following standard well-known result (see [3, 12]). In order to state the results, we need the following definition (see [8]).

Definition 2.1. *Let* m *be a nonnegative integer. We say that the initial condition* $(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ *satisfies the compatibility condition of the* m*-th order relative to the problem* $(1.1) - (1.3)$ *if*

$$u_k \in H^{m+1-k}(\Omega) \cap H_0^1(\Omega), \quad k = 0, 1, 2, ..., m \quad and \quad u_{m+1} \in L^2(\Omega),$$

where $\{u_k\}$ *is defined inductively by*

$$u_{k+2} = \triangle u_k - a(x)u_{k+1}, \quad k = 0, 1, ..., m-1.$$

Theorem 2.1. *Let* $a(\cdot) \in C^{m-1}(\overline{\Omega} \cap B_L)$ *and* $(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ *satisfy the compatibility condition of the* m*-th order relative to the problem* $(1.1) - (1.3)$. *Then there exists a unique solution* $u(t)$ *to the problem* $(1.1) - (1.3)$ *such that*

$$u \in X^m \equiv \cap_{k=0}^m C^k([0, \infty); H^{m+1-k}(\Omega) \cap H_0^1(\Omega)) \cap C^{m+1}([0, \infty); L^2(\Omega)),$$

and the linear mapping

$$(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega) \mapsto u \in X^m$$

is continuous.

The main result of this paper reads as follows:

Theorem 2.2. *Let m be a positive integer. Let $(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfy the compatibility of the m-th order relative to the problem $(1.1) - (1.3)$. If $a(\cdot) \in C^{m-1}(\bar{\Omega})$ satisfies (1.6) for some $0 < p < 1$ in Hypothesis A, then we have the decay estimates:*

If $N \geq 2m$ and $N - 2m \leq mp$, then we have

$$E_{loc}^{L+\varepsilon t}(t) = \mathcal{O}\left(t^{-\frac{2mp}{2mp+N}}\right)$$

for the solution $u(t) \in X^m$ and any $0 < \varepsilon < 1$.

Remark 2.1. *If $1 \leq N < 2m$, then Nakao [9] prove that*

$$E_{loc}^{L+\varepsilon t}(t) = \mathcal{O}\left(t^{-\frac{2mp}{2mp+N}}\right)$$

Remark 2.2. *We can observe that the decay rates $-2m(p+1)/(2mp+N)$ and $-2m/(2mp+N)$ tend to $(p+1)/p > 1$ and $1/p > 1$ as m tends to infinity.*

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