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BILINEAR BOUNDARY OPTIMAL CONTROL OF THE VELOCITY TERM IN A KIRCHHOFF PLATE EQUATION

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ABSTRACT. In this paper, we consider a bilinear boundary optimal control problem for a Kirchhoff plate equation. The control is a function of the spatial variables and acts a multiplier of the velocity term. We prove the existence of a boundary optimal for a given objective functional and uniqueness of this optimal control for T sufficiently small.

1. Introduction

In this paper, we consider the boundary control problem of a Kirchhoff plate equation

$$(1.1)\quad \begin{cases} w_{tt} - \Delta w_{tt} + \Delta^2 w = 0 \text{ in } Q = \Omega \times (0,T), \\ w(0) = w_0, w_t(0) = w_1 \text{ in } \Omega, \\ w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Sigma_0 = \Gamma_0 \times (0,T), \\ \Delta w + (1-\mu) B_1 w = 0 \text{ on } \Sigma_1 = \Gamma_1 \times (0,T), \\ \frac{\partial}{\partial \nu}(\Delta w) + (1-\mu) B_2 w - \frac{\partial w_{tt}}{\partial \nu} = h(x,y) w_t + g(w) \text{ on } \Sigma_1 = \Gamma_1 \times (0,T), \end{cases}$$

where the control set

$$h \in U_M = \{h \in L^\infty(\Gamma_1) : -M \le h(x,y) \le M\},$$

$\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary $\Gamma := \partial\Omega$, $\Gamma_0 \cup \Gamma_1 = \Gamma, \Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \ne \emptyset$, $\nu = (\nu_1, \nu_2)$ is the outward unit normal vector on Γ and

$$B_1 w = 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx},$$
$$B_2 w = \frac{\partial}{\partial \tau}[(\nu_1^2 - \nu_2^2) w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})],$$

where τ is the tangential direction along Γ_1. The constant μ, $0 < \mu < 1/2$, represents Poisson's ratio and the function g is continuously differentiable real-valued function with satisfies the following conditions :

(H_1) $g(0) = 0$, $0 < g'(s) \le L$, where L is a positive constant.

We take as our objective functional

$$J(h) = \frac{1}{2}\int_Q (w-z)^2 dQ + \frac{\beta}{2}\int_{\Gamma_1} h(x,y)^2 d\Gamma,$$

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where $z \in L^2(Q)$ is a given target function and the quadratic term in h represents the cost of implementing the control with weighting factor $\beta > 0$. We seek to minimize the objective functional and characterize an optimal control $h^* \in U_M$ such that

$$J(h^*) = \min_{h \in U_M} J(h).$$

For convenience, we assume that

$$z \in C([0,T]; L^2(\Omega)), \;\; z_t \in C([0,T]; L^2(\Omega)).$$

For background on plate models and control, see the books by Lagnese and Lions [8], Lagnese [7], Kormornik [5], Li and Yong [10]. The bilinear control case treated here does not fit into the Riccati framework [9], even though the objective functional is quadratic, the state equation has a bilinear term, hw_t. Bilinear control problem was introduced in Bradley and Lenhart [3] with control acting through the term hw. Recently, Bradley, Lenhart and Yong [4] and Bradley and Lenhart [3] were considered the bilinear optimal control of the velocity term in a Kirchhoff plate equation. The author M.E. Bradley [2] studied the exponential stabilization for a Kirchhoff plate with boundary nonlinearities. In contrast to the problem of a Kirchhoff plate with boundary nonlinearities, we studied the boundary control problem for a Kirchhoff plate equation. It is important to observe that it has never been considered boundary optimal control problem for a Kirchhoff plate equation in the literature as in the present paper. Our paper is organized as follows. In Section 2, we show well-posedness of our state problem. In Section 3, we show the existence of an optimal control by a minimizing sequence argument. In Section 4, we derive a characterization for optimal controls, in terms of the solutions of an optimality system. In Section 5, we prove that the optimal control is unique for small time T, provided that initial data are taken to be sufficiently smooth.

2. Well-posedness of the state equation

We will begin by proving existence, uniqueness, and regularity results for the state equation. We first define solution spaces :

$$H^2_{\Gamma_0}(\Omega) = \{w \in H^2(\Omega) : w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\}$$

and

$$\mathcal{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega).$$

Note that the bilinear form on $H^2_{\Gamma_0}(\Omega)$

$$a(u,v) = \int_\Omega [\Delta u \Delta v + (1-\mu)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] d\Omega$$

induces a norm on $H^2_{\Gamma_0}(\Omega)$ which is equivalent to the usual $H^2(\Omega)$ norm on $H^2_{\Gamma_0}(\Omega)$ (see [6]). Throughout this paper we denote

$$(u,v) = \int_\Omega u(x,y)v(x,y) d\Omega, \;\; ||u||^2 = \int_\Omega u(x,y)^2 d\Omega,$$

and

$$(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x,y)v(x,y)d\Gamma.$$

Moreover, we denote $||\cdot||_X$ the norm on a Banach space X.

Definition 2.1 Given $h \in U_M$, $\tilde{w} = \tilde{w}(h) = (w(h), w_t(h))$ is a weak solution to (1.1) if $\tilde{w} \in C([0,T];\mathcal{H}), \tilde{w}(0) = (w_0, w_1)$ and $\tilde{w}$ satisfies

$$\begin{aligned}
&\int_0^T \big\{\langle w_{tt}, \phi\rangle + (\nabla w_{tt}, \nabla\phi) + a(w,\phi)\big\}dt \\
&= \int_0^T \int_{\Gamma_1} g(w)\phi d\Gamma dt + \int_0^T \int_{\Gamma_1} h w_t \phi d\Gamma dt \ \ \forall \phi \in H^2_{\Gamma_0}(\Omega), \\
&w(0) = w_0, w_t(0) = w_1,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^2_{\Gamma_0}(\Omega)$ and $[H^2_{\Gamma_0}(\Omega)]'$.

Theorem 2.1. *(i) Let $\tilde{w}(0) = (w_0, w_1) \in \mathcal{H}$ and $h \in U_M$. Then the system (1.1) has a unique weak solution.*
(ii) In addition, if $(w_0, w_1) \in D_0$, where

$$\begin{aligned}
D_0 = \big\{&(w_0,w_1) \in (H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega) : \Delta w_0 + (1-\mu)B_1 w_0 = 0 \ \text{on } \Gamma_1, \\
&\frac{\partial}{\partial\nu}(\Delta w_0) + (1-\mu)B_2 w_0 - \frac{\partial w_{tt}(0)}{\partial\nu} = g(w_0) + hw_1 \ \text{on } \Gamma_1\big\} \ \text{for } h \in U_M,
\end{aligned}$$

then the weak solution satisfies

$$\tilde{w} \in C([0,T];(H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega)) \ \text{and } w_{tt} \in C([0,T];L^2(\Omega)).$$

Lemma 2.1. *(A priori estimate). Given $\tilde{w}(0) \in \mathcal{H}$ and $h \in U_M$, the weak solution to (1.1) satisfies*

$$||\tilde{w}||_{C([0,T];\mathcal{H})} \le ||\tilde{w}(0)||_{\mathcal{H}}(1+2MT)^{1/2}e^{c_3 MT}, \tag{2.1}$$

where c_3 is a constant.

Theorem 2.2. *There exists an optimal control $h^* \in U_M$, which minimizes the objective functional $J(h)$ over h in U_M.*

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