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UNIFORM DECAY FOR A HYPERBOLIC SYSTEM WITH DIFFERENTIAL INCLUSION AND NONLINEAR MEMORY SOURCE TERM ON THE BOUNDARY

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Abstract. We prove the existence and uniform decay rates of global solutions for a hyperbolic system with a discontinuous and nonlinear multi-valued term and a nonlinear memory source term on the boundary.

1. Introduction

In this paper we are concerned with the existence and uniform decay rates of the solutions for a hyperbolic system with differential inclusion and memory source term on the boundary of the form

$$u'' - div(a\nabla u) + |u|^\gamma u = 0 \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_1 \times (0,\infty), \tag{1.2}$$

$$(a\nabla u)\cdot \nu + u' + \Xi = \int_0^t h(t-\tau) f(u(\tau))d\tau \text{ on } \Gamma_0 \times (0,\infty), \tag{1.3}$$

$$u(x,0) = u_0, \ \ u'(x,0) = u_1 \text{ in } \Omega, \tag{1.4}$$

$$\Xi \in \varphi(u(x,t)) \text{ a.e. } (x,t) \in \Gamma_0 \times (0,\infty), \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 2)$ with sufficiently smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1, \bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 have positive measures, $u' = \frac{\partial u}{\partial t}, u'' = \frac{\partial^2 u}{\partial t^2}$, $a \in C^1(\bar{\Omega})$, f is a nonlinear function, ν is the unit outward normal to Γ and φ is a discontinuous and nonlinear set valued mapping by filling in jumps of a function $b \in L^\infty_{loc}(\mathbb{R})$. In the rest of the paper let us assume that

$$\frac{2}{n-1} < \gamma \leq \frac{2}{n-2} \quad \text{if } n \geq 3,$$

and

$$\gamma > 2 \quad \text{if } n = 2.$$

The precise hypotheses on the above system will be given in the next section. Recently, a class of viscoelastic problems are studied by many authors [2,3,10,13]. M. Aassila [1] investigated the global existence of a solution to a system (1.1) and (1.4) with damping terms and the Dirichlet boundary conditions when $a(x) \equiv 1$.

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M.M. Cavalcanti et al. [3] studied the existence and uniform decay of solutions of the damped semilinear viscoelastic wave equation with the Dirichlet boundary conditions of the form

$$\begin{cases} u'' - \Delta u + \alpha u + \beta |u'|^\rho u' + \delta |u|^\rho u + \int_0^t h(t-\tau)\Delta u(\tau)d\tau = 0 \text{ in } \Omega \times (0,\infty), \\ u(x,0) = u_0, \ u'(x,0) = u_1 \text{ in } \Omega, \end{cases}$$

where Ω is any bounded or finite measure domain of $\mathbb{R}^n$ and the constants α, β, ρ and δ are positive and satisfy some conditions. Motivated by works of them, we consider more generalized problems (1.1)-(1.5) with a discontinuous and nonlinear multi-valued term φ and a nonlinear memory source term on the boundary. The background of these variational problems are in physics, especially in solid mechanics, where non-monotone and multi-valued constitutive laws lead to differential inclusion. We refer to [5,11,12] to see the applications of such differential inclusions. In this paper we prove the existence of solutions of the variational inequality problems (1.1)-(1.5). Moreover the uniform decay of the energy

$$E(t) = \frac{1}{2}||u'(t)||^2 + \frac{1}{2}\int_\Omega a(x)|\nabla u(x,t)|^2 dx + \frac{1}{\gamma+2}||u(t)||_{\gamma+2}^{\gamma+2}$$

is proved by assuming that μ (see assumption $(A_2)^*$ below) is sufficiently small and the kernel h in the memory term decays exponentially. At this point it is important to mention that such differential inclusions were studied by some authors [4,8,9,14,15], but, as far as we are concerned it has never been considered differential inclusion acting on the boundary and no decay rates were obtained as in this present paper in the literature. Our paper is organized as follows : In Section 2, we give assumptions and state the main results. In Section 3, we prove the existence of solution of the problems (1.1)-(1.5) by using the Faedo-Galerkin method. Finally, in Section 4, we prove the uniform decay of energy by using the Liapunov functional developed by Kormornik and Zuazua [6].

2. Assumptions and main results

Throughout this paper we denote

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}, \ (u,v) = \int_\Omega u(x)v(x)dx,$$

$$(u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma, \ ||u||_{2,\Gamma_0}^2 = \int_{\Gamma_0} |u(x)|^2 d\Gamma.$$

Let us denote V^* by the dual space of V, $||\cdot||_*$ the norm of V^* and $\langle \cdot, \cdot \rangle$ the dual pairing between V and V^*. For simplicity, we denote $||\cdot||_{L^2(\Omega)}, ||\cdot||_{L^p(\Omega)} (1 \le p \le \infty)$ and $||\cdot||_{2,\Gamma_0}$ by $||\cdot||, ||\cdot||_p$ and $||\cdot||_{\Gamma_0}$, respectively. Let λ_0 and λ be the smallest positive constants such that

$$||u||^2 \le \lambda_0 ||\nabla u||^2, \ ||u||_{\Gamma_0}^2 \le \lambda ||\nabla u||^2, \ \forall u \in V. \tag{2.1}$$

We formulate the following assumptions :

$(\mathbf{A_1})$ Assumptions on $\mathbf{a}$

Let $a \in C^1(\bar{\Omega})$ satisfying $a(x) \ge a_0 > 0$ in Ω for some a_0.

For short notation, define $a(u,v) = \sum_{j=1}^{n} \int_\Omega a(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx$. By the above assumption on a, we have

$$a_0 ||\nabla u||^2 \leq a(u,u) \leq a_1 ||\nabla u||^2, \;\; \forall u \in V \text{ for some } a_1 > 0.$$

($\mathbf{A_2}$) Assumptions on **b**
Let $b : \mathbb{R} \to \mathbb{R}$ be a locally bounded function satisfying

$$|b(s)| \leq \mu_0(1 + |s|) \;\; \forall s \in \mathbb{R} \text{ for some } \mu_0 > 0.$$

In order to get the uniform decay rates for the solutions of problems (1.1)-(1.5) we shall use the following stronger hypothesis:
$(\mathbf{A_2})^*$ $|b(s)| \leq \mu|s|$ and $b(s)s \geq \mu_1 s^2$, where $\mu_1 > 0$ and $0 < \mu < 1$.
The multi-valued function $\varphi : \mathbb{R} \to \mathbb{R}$ is obtained by filling in jumps of a function $b : \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\epsilon, \overline{b}_\epsilon, \underline{b}, \overline{b} : \mathbb{R} \to \mathbb{R}$ as follows

$$\begin{aligned}
&\underline{b}_\epsilon(t) = ess \;\; inf_{|s-t|\leq\epsilon} \;\; b(s), \;\; \overline{b}_\epsilon(t) = ess \;\; sup_{|s-t|\leq\epsilon} \;\; b(s);\\
&\underline{b}(t) = \lim_{\epsilon \to 0^+} \underline{b}_\epsilon(t), \;\; \overline{b}(t) = \lim_{\epsilon \to 0^+} \overline{b}_\epsilon(t);\\
&\varphi(t) = [\underline{b}(t), \overline{b}(t)].
\end{aligned}$$

We shall need a regularization of b defined by

$$b^m(t) = m \int_{-\infty}^{\infty} b(t-\tau)\rho(m\tau)d\tau,$$

where $\rho \in C_0^\infty((-1,1)), \rho \geq 0$ and $\int_{-1}^{1} \rho(\tau)d\tau = 1$.

Remark 2.1. *It is easy to show that* b^m *is continuous for all* $m \in \mathbb{N}$ *and* $\underline{b}_\epsilon, \overline{b}_\epsilon, \underline{b}, \overline{b}, b^m$ *satisfy the same condition* (A_2) *or* $(A_2)^*$ *with possibly different constants if* b *satisfies* (A_2) *or* $(A_2)^*$*. So, in the sequel, we denote the different constants by the same symbols as original constants.*

($\mathbf{A_3}$) Assumptions on **f**
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$|f(s)| \leq \alpha(1 + |s|), \;\; \forall s \in \mathbb{R}$$

for some positive constant α.
($\mathbf{A_4}$) Assumptions on the kernel **h**
Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable function verifying

$$-\xi_1 h(t) \leq h'(t) \leq -\xi_2 h(t), \;\; \forall t \geq t_0$$

for some $t_0 > 0, h(0) = 0$ and $1 - \frac{\lambda}{a_0} \int_0^\infty h(s)ds = l > 0$.

Definition A function $u(x,t)$ is a solution to problems (1.1)-(1.5) if for every $T > 0$, u satisfies

$$u \in L^\infty(0,T;V), u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^2(\Gamma_0)), u'' \in L^2(0,T;V^*),$$

there exists $\Xi \in L^2(0,T;L^2(\Gamma_0))$ and the following relations hold:

$$\int_0^T \{\langle u'', v\rangle + a(u(t),v) + (|u(t)|^\gamma u(t), v) + (u'(t),v)_{\Gamma_0} + (\Xi, v)_{\Gamma_0}\}dt$$
$$= \int_0^T \int_0^t h(t-\tau)(f(u(\tau)),v)_{\Gamma_0} d\tau dt, \ \ \forall v \in V,$$
$$\Xi(x,t) \in \varphi(u(x,t)) \ \text{ a.e. } \ (x,t) \in \Gamma_0 \times (0,T),$$
$$u(x,0) = u_0, \ \ u'(x,0) = u_1 \text{ on } \Omega.$$

Now we are in a positon to state our results.

Theorem 2.1. *Assume the conditions* $(A_1)-(A_4)$ *hold. Then for every* $(u_0,u_1) \in V \times L^2(\Omega)$ *there exists a solution of problems (1.1)-(1.5).*

Theorem 2.2. *Assume the conditions* $(A_1),(A_2)^*,(A_3)$ *and* (A_4) *hold and* $(u_0,u_1) \in V \times L^2(\Omega)$*. Furthermore, if we assume* $\frac{||\nabla a||_\infty}{a_0} \le \mu$ *and consider* $||h||_{L^1(0,\infty)}$ *and* μ *(given in* $(A_2)^*$*) sufficiently small, the energy determined by the soluions of problems (1.1)-(1.5) decays exponentially, that is,*

$$E(t) \le C_3 \exp(-\frac{2}{3}C_2 t) \quad a.e. \ \ t \ge t_0,$$

for some positive constants C_2 *and* C_3*.*

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