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GLOBAL EXISTENCE AND UNIFORM DECAY FOR THE COUPLED KLEIN-GORDON-SCHRÖDINGER EQUATIONS WITH NONLINEAR BOUNDARY DAMPING AND MEMORY TERM

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ABSTRACT. In this paper we are concerned with the existence and energy decay of solution to the initial boundary value problem for the coupled Klein-Gordon-Schrödinger equations with nonlinear boundary damping and memory term.

1. INTRODUCTION

In this paper, we are concerned with the existence and uniform decay of the solutions for the coupled Klein-Gordon-Schrödinger equations with nonlinear boundary damping and memory term of the form:

$$i\psi_t + \Delta\psi + i|\psi|^2\psi + i\alpha\psi = -\phi\psi \quad \text{in} \quad Q = \Omega \times (0,\infty), \tag{1.1}$$

$$\phi_{tt} - \Delta\phi + \mu^2\phi + \beta\phi_t = |\psi|^2 \quad \text{in} \quad Q = \Omega \times (0,\infty), \tag{1.2}$$

$$\psi = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0,\infty), \tag{1.3}$$

$$\phi = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty), \tag{1.4}$$

$$\frac{\partial\phi}{\partial\nu} + \phi + \phi' + g(t)|\phi'|^\rho\phi' = g * |\phi|^\gamma\phi \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty), \tag{1.5}$$

$$\psi(x,0) = \psi^0(x), \quad \phi(x,0) = \phi^0(x), \quad \phi_t(x,0) = \phi^1(x) \quad \text{on} \quad x \in \Omega, \tag{1.6}$$

where Ω be a bounded domain of $\mathbb{R}^n$, $n \le 3$, with C^2 boundary Γ. We divide the boundary into two parts $\Gamma = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measures. Let Q be the infinite cylinder $\Omega \times (0,\infty)$ whose lateral boundary is $\Sigma = \Gamma \times (0,\infty)$, $g * u = \int_0^t g(t-r)u(r)dr$ and ν denotes the unit outer normal vector pointing towards Ω. Assuming the kernel g provides a damping effect, we prove existence of strong solution $u = u(x,t)$.

The above equations describe a generalization of the classical model of the Yukawa interaction of conserved complex nucleon field with neutral real meson field. Here, ψ is complex scalar nucleon field while ϕ is a real scalar meson one, and the real constant μ describes the mass of a meson.

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In I.Fukuda and M.Tsutsumi[4,5,6], A.Bachelot[1], J.B.Baillon and J.M.Chadam[2] and N.Hayashi and W.Von Wahl[8], the unique global existence for the Cauchy problem to

$$\begin{aligned} &i\psi_t + \frac{1}{2}\Delta\psi = -\phi\psi \\ &\phi_{tt} - \Delta\phi + \mu^2\phi = |\psi|^2 \end{aligned}$$

is established and some conservation laws are verified.

In M.M.Cavalcanti, V.N.Domingos Cavalcanti[3], the existence, uniqueness and uniform decay for the solutions of the coupled Klein-Gordon-Schrödinger equations of the form:

$$\begin{aligned} &i\psi_t + \Delta\psi + \mu^2|\psi|^2\psi + i\gamma\psi = -\phi\psi \;\text{ in }\; \Omega\times(0,\infty), \\ &\phi_{tt} - \Delta\phi + \mu^2\phi + F(\phi,\phi_t) = \beta|\psi|^{2\theta} \;\text{ in }\; \Omega\times(0,\infty), \\ &\psi = \phi = 0 \;\text{ on }\; \Gamma\times(0,\infty), \\ &\psi(x,0) = \psi^0(x), \;\; \phi(x,0) = \phi^0(x), \;\; \phi_t(x,0) = \phi^1(x) \;\text{ in }\; \Omega. \end{aligned}$$

To obtain the existence of solutions we make use of Faedo-Galerkin's approximation and also to show the uniform stabilization we use the perturbed energy method. Our paper is organized as follows: In Section 2, we give some notations, assumptions and main result. In Section 3, we prove the existence of solutions of the problems (1.1)-(1.6) and the uniform decay of energy is given in Section 4.

2. Notaions and main results

Throughout this paper we denote

$$V = \{v \in H^1(\Omega) : v = 0 \;\text{ on }\; \Gamma_0\}, \;\; (u,v) = \int_\Omega u(x)\overline{v(x)}dx, \;\; (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)\overline{v(x)}d\Gamma,$$

$$\|u\|_{p,\Gamma_1} = (\int_{\Gamma_1} |u(x)|^p d\Gamma)^{\frac{1}{p}} \;\text{ and }\; \|u\|_\infty = \|u(t)\|_{L^\infty(\Omega)}.$$

Now, we state the general assumptions.

(A) Assumption on the kernel g

Let us consider the function

$$g \in W^{1,\infty}(0,\infty)\cap W^{1,1}(0,\infty) \;\text{ such that }\; g(t) \geq 0, \;\; \forall t \geq 0 \tag{2.1}$$

and there exists a positive constant t_0 such that

$$\begin{cases} -m_0 g(t) \leq g'(t) \leq -m_1 g(t), \;\; \forall t \geq t_0, \\ g(0) = 0, \;\; |g'(t)| \leq m_2 g(t), \;\; \forall t \in [0,t_0] \end{cases} \tag{2.2}$$

for some m_0, m_1, $m_2 > 0$, $m_1 > 2(\gamma+2)$ and

$$l = 1 - \int_0^\infty g(r)dr > 0. \tag{2.3}$$

We define the energy of system (1.1)-(1.6) by

$$E(t) = \frac{1}{2}\|\psi(t)\|^2 + \frac{1}{2}\|\phi'(t)\|^2 + \frac{1}{2}\|\nabla\phi(t)\|^2 + \frac{\mu^2}{2}\|\phi(t)\|^2 + \frac{1}{2}\|\phi(t)\|^2_{\Gamma_1}.$$

Lemma 2.1. (Gronwall's inequality)

Let $\xi(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies for t a.e., the integral inequality $\xi(t) \leq M_1 \int_0^t \xi(t)ds + M_2$ for constant $M_1, \quad M_2 \geq 0$. Then $\xi(t) \leq M_2(1 + M_1 te^{M_1 t})$, for $0 \leq t \leq T$, a.e.

Proof. See, e.g.[9]. □

Lemma 2.2. (Generalized Hölder inequality)

Let Ω be an open set in $\mathbb{R}^n$ and $1 \leq p_1, p_2, \cdots, p_n \leq \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$ and assume $u_k \in L^{p_k}(\Omega)$, for $k = 1, 2, \cdots, m$. Then

$$\int_\Omega |u_1 \cdot u_2 \cdots u_m| dx \leq \Pi_{k=1}^m \|u_k\|_{p_k}.$$

Proof. See, e.g.[7]. □

Now, we are in a position to state our main result:

Theorem 2.1. *$\{\psi^0, \phi^0, \phi^1\} \in \{H_0^1(\Omega) \cap H^2(\Omega)\} \times \{V \cap H^2(\Omega)\} \times L^2(\Omega)$ satisfy $\frac{\partial \phi_0}{\partial \nu} + \phi_0 + \phi_1 = 0$. Under assumptions (A), suppose that γ, ρ with $\rho \geq \gamma$ and $\gamma, \rho > 0$ if $n = 1, 2$ or $0 < \gamma, \rho \leq \frac{1}{n-2}$ if $n = 3$ and $\alpha > \frac{1}{4}, \beta > 4$. Then problem (1.1)-(1.6) has a unique strong solution verifying*

$$\psi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \quad \psi' \in L^\infty(0, \infty; L^2(\Omega)),$$
$$\phi \in L^\infty(0, \infty; V), \quad \phi' \in L^\infty(0, \infty; V), \quad \phi'' \in L^\infty(0, \infty; L^2(\Omega)).$$

Moreover, if $\|g\|_{L^1(0,\infty)}$ is sufficiently small, we have the following energy decay

$$E(t) \leq 3l^{-1} E(0) exp(-\frac{\varepsilon}{2} C_2 t), \quad \forall t \geq t_0 \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_0],$$

where C_2 and ε_0 are positive constants.

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