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CONTROLLABILITY FOR THE SECOND ORDER DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. The purpose of this paper is to investigate the control problem for the second order differential equations with nonlocal state initial term and nonlocal damping initial term on a Banach space.

1. Introduction

In this paper, we establish sufficient conditions for the controllability of second-order differential equations in Banach spaces with nonlocal state initial term and nonlocal damping initial term. More precisely, we consider the following semilinear system:

$$\text{(1.1)} \qquad \begin{cases} \dfrac{d^2x(t)}{dt^2} = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in I = [0, a], \\ x(0) = x_0 + p(x), \\ \dot{x}(0) = x_1 + q(x), \end{cases}$$

where A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t),\ t \in R\}$ in a Banach space X, $f : R \times X \to X$, $p,\ q : C(I : X) \to X$ are given continous nonlinear functions, B is a bounded linear operator on U. Also, u is control function on U.

Recently, Hernandez([5]) is established the existence of mild and classical solutions for the equation (1.1)$(B = 0)$. Kang etal.([6]) are introduced the new type control function for the equation (1.1)$(p = 0, q = 0)$.

2. Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas.

Definition 2.1.[7,8] A one parameter family $C(t), t \in R$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family iff
(2.1) $C(s + t) + C(s - t) = 2C(s)C(t)$ for all $s,\ t \in R$,
(2.2) $C(0) = I$,
(2.3) $C(t)x$ is continuous in t on R for each fixed $x \in X$.

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If $C(t),\ t \in R$ is a strongly continuous cosine family in X, then $S(t),\ t \in R$ is the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)xds,\ x \in X,\ t \in R. \tag{2.4}$$

The infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in R$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(0)x.$$

where $D(A) = \{x \in X : C(t)x$ is a twice continuously differentiable function of $t\}$.

We shall also make use of the set

$E = \{x \in X : C(t)x$ is a once continuously differentiable function of $t\}$.

Lemma 2.1.[7,8] Let $C(t),\ t \in R$ be a strongly continuous cosine family in X. The following are true:

(2.5) $C(t) = C(-t)$ for all $t \in R$,

(2.6) $C(s), S(s), C(t)$ and $S(t)$ commute for all $s, t \in R$,

(2.7) $S(t)x$ is continuous in t on R for each fixed $x \in X$,

(2.8) there exist constants $K \geq 1$ and $\omega \geq 0$ such that

$$|C(t)| \leq Ke^{\omega|t|} \text{ for all } t \in R,\ |S(t_1) - S(t_2)| \leq K\left|\int_{t_2}^{t_1} e^{\omega|s|}ds\right| \text{ for all } t_1, t_2 \in R,$$

(2.9) if $x \in E$, then $S(t)x \in D(A)$ and $\frac{d}{dt}C(t)x = AS(t)x$,

(2.10) if $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$.

Lemma 2.2.[7,8] Let $C(t), t \in R$ be a strongly continuous cosine family in X with infinitesimal generator A. If $g : R \to X$ is continuously differentiable, $x_0 \in D(A), y_0 \in E$, and

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)g(s)ds, \quad t \in R,$$

then $w(t) \in D(A)$ for $t \in R, w$ is twice continuously differentiable, and w satisfies

$$\frac{d^2w(t)}{dt^2} = Aw(t) + g(t), \quad t \in R, \quad w(0) = x_0,\ \dot{w}(0) = y_0. \tag{2.11}$$

Conversely, if $g : R \to X$ is continuous, $w(t) : R \to X$ is twice continuously differentiable, $w(t) \in D(A)$ for $t \in R$, and w satisfies (2.11), then

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)g(s)ds, \quad t \in R.$$

We shall also make use of the set

$$E = \{x \in X : C(t)x \text{ is a once continuously differentiable function of } t\}.$$

Lemma 2.3 [5]. (Banach Fixed Point Theorem) Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has Precisely one fixed point.

3. Controllability

This section is concerned with the controllability for the nonlinear second order control system (1.1).

For the (1.1), a integral equation can be written as

$$\begin{cases} x(t) = & C(t)(x_0 + p(x)) + S(t)(y_0 + q(x)) \\ & + \int_0^t S(t-s)f(s, x(s))ds + \int_0^t S(t-s)Bu(s)ds, \\ x(0) = & x_0 + p(x), \quad \dot{x}(0) = y_0 + q(x). \end{cases} \tag{3.1}$$

We define the controllability concept for the nonlinear second order control system.

Definition 3.1 The system (1.1) is said to be nonlocal controllable on $[0, T]$ if for every x_0, $x^1 \in D(A)$ and y_0, $y^1 \in E$ there exists a control $u \in L^2([0, T] : U)$ such that the solution $x(\cdot)$ of (1.1) satisfies $x(T) = x^1 + p(x) \in D(A)$ and $\dot{x}(T) = y^1 + q(x) \in E$, where x^1 is state target and y^1 is dampping target.

We assume the following hypotheses:

- (H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$, of bounded linear operator in the Banach space X. There exists constant $M \geq 1$ such that
$$|C(t)| \leq Ke^{\omega|t|} \leq M \quad \text{for all} \quad t \in [0, T].$$
- (H2) The associated sine family $S(t), t \in R$, is compact with $|S(t)| \leq N$.
- (H3) The nonlinear operator $f : [0, T] \times X \to X$ satisfies a global Lipschitz condition, i.e., there exists a finite constant $k > 0$ such that
$$|f(t, x(t)) - f(t, y(t))| \leq k|x(t) - y(t)|.$$
- (H4) For all $x, y \in C([0, T] : X)$, there exists $l_p > 0$ such that
$$|p(x) - p(y)| \leq l_p|x - y|_T$$
where $|x|_T = \sup_{0 \leq t \leq T} |x(t)|$. And $p(0) = 0$.
- (H5) For all $x, y \in C([0, T] : X)$, there exists $l_q > 0$ such that
$$|q(x) - q(y)| \leq l_q|x - y|_T$$
where $|x|_T = \sup_{0 \leq t \leq T} |x(t)|$. And $q(0) = 0$.
- (H6) There exists $L > 0$ such that
$$\sup_{0 \leq t \leq T} |AS(t)| \leq L$$
.
- (H7) The linear operator $G_1 : U \to X$ defined by
$$G_1 u = \int_0^T S(T - s)Bu(s)ds$$

and there exists an bounded invertible operator ${G_1}^{-1}$ defined on $L^2([0,T]:U)/\mathrm{ker}G_1$.

- (H8) The linear operator $G_2 : U \to X$ defined by

$$G_2 u = \int_0^T C(T-s)Bu(s)ds$$

and there exists an bounded invertible operator ${G_2}^{-1}$ defined on $L^2([0,T]:U)/\mathrm{ker}G_2$ and Bu is continuously differentiable.

- (H9) For sufficiently small $T > 0$, $Ml_p + Nl_q + NkT < 1$ and $Ll_p + Ml_q + MkT < 1$.

$C = C([0,T]:X)$ and $C^1 = C^1([0,T]:X)$ are the Banach space of continuous X valued functions on $[0,T]$ and is endowed with the supremum norm. Now, we prove the result of controllability for the control system (1.1) with above assumptions.

Theorem 3.1. If the hypotheses (H1)-(H9) are satisfied, then the system (1.1) is nonlocal controllable on $[0,T]$.

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