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UNIFORM DECAY FOR THE KIRCHHOFF PLATE EQUATIONS WITH NONLINEAR DISSIPATION

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ABSTRACT. This paper is concerned with the existence of solutions for the Kirchhoff plate equations with a memory condition at the boundary. We show that the exponential decay to the solution, provided the relaxation functions also decays exponentially.

1. INTRODUCTION

We consider the following Kirchhoff plate equations with nonlinear dissipation:

$$u_{tt} + \Delta^2 u + \rho(x, u_t) = 0, \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma_0 \times (0, \infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_1(t-s)(\mathcal{B}_1 u(s) + \gamma_1 \frac{\partial u}{\partial \nu(s)})ds = 0, \text{ on } \Gamma_1 \times (0, \infty), \tag{1.3}$$

$$-u + \int_0^t g_2(t-s)(\mathcal{B}_2 u(s) - \gamma_2 u(s))ds = 0, \text{ on } \Gamma_1 \times (0, \infty), \tag{1.4}$$

$$u(0, x) = u_0(x), \; u_t(0, x) = u_1(x), \text{ in } \Omega, \tag{1.5}$$

where Ω be a open bounded set of R^2 with regular boundary Γ. We divide the boundary into two parts:

$$\Gamma = \Gamma_0 \cup \Gamma_1 \quad \text{with} \quad \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset; \text{ and } \Gamma_0 \neq \emptyset. \tag{1.6}$$

Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ, and let us denotes by $\eta = (-\nu_2, \nu_1)$ the unit tangent positively oriented on Γ.

$$\mathcal{B}_1 u = \Delta u + (1-\mu) B_1 u,$$
$$\mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

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and

$$B_1 u = 2\nu_1\nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2},$$
$$B_2 u = (\nu_1^2 - \nu_2^2)\frac{\partial^2 u}{\partial x \partial y} + \nu_1\nu_2 (\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}).$$

In (1.1), u denote the position of the plate. In (1.3) and (1.4) the relaxation functions $g_1,\ g_2 \in C^1(0,\infty)$ are positive and nondecreasing with $g_1(0) > 0,\ g_2(0) > 0$ and γ_1, γ_2 are small positive constants. This system models the small transversal vibrations of a thin plate whose Poisson coefficient is equal to μ, with $\mu \in]0, \frac{1}{2}[$; see e. g. Lagnese [2] or Lagnese and Lions [4]. We assume that there exists $x_0 \in R^2$ such that

$$\Gamma_0 = \{x \in \Gamma : \nu(x)\cdot(x - x_0) \le 0\}, \tag{1.7}$$

$$\Gamma_1 = \{x \in \Gamma : \nu(x)\cdot(x - x_0) > 0\}. \tag{1.8}$$

If we denote by $m(x) = x - x_0$, the compactness of Γ_1 and condition (1.8) imply that there exist a small positive constant δ_0 such that

$$0 < \delta_0 \le m(x)\cdot\nu(x),\ \forall x \in \Gamma_1. \tag{1.9}$$

The uniform stabilization of Kirchhoff plates equations with linear or nonlinear boundary feedback was investigated by several authors, see for example [2, 9, 10] among others. In all the above works was proved also existence and uniqueness of solution. Recently, Santos and Junior [10] investigated the stability of solutions for Kirchhoff plate equations with boundary memory condition. The main feature which distinguishes this paper from other related works is the fact that the effect of the dissipative term u_t and the boundary condition is of the memory type. Our main result is show that the solution of system (1.1)-(1.5) decays uniformly in time, with rates depending on the rate of decay of the relaxation functions. More precisely, depending by k_1 and k_2 the resolvents kernels of g_1' and g_2', respectively, we show that the solution decays exponentially to zero provided k_1 and k_2 decays exponentially to zero.

The method used here is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t) + c_2 e^{-\gamma t}$$

for some positive constants $c_1,\ c_2$ and γ. Because of condition (1.2) the solution of the system (1.1)-(1.5) must belong to the following space:

$$W = \{w \in H^2(\Omega); w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\}.$$

Let us define the bilinear form $a(\cdot,\cdot)$ as follows:

$$a(u,v) = \int_\Omega \{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + \mu(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2})$$
$$+ 2(1-\mu)\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\}dxdy.$$

The notation used in this paper is standard and can be found in Lions's book [4]. In the sequel by c (sometimes $c_1,\ c_2,\cdots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In Section 2 we establish a existence and regularity result. In Section 3 we prove the uniform rate of exponential decay.

We finish this section with the following Lemma which will be useful in what follows:

Lemma 1.1. *Let u and v be functions in $H^4(\Omega)\cap W$. Then we have*

$$\int_\Omega (\Delta^2 u)v dxdy = a(u,v) + \int_{\Gamma_1}\{(\mathcal{B}_2 u)v - (\mathcal{B}_1 u)\frac{\partial v}{\partial \nu}\}d\Gamma_1. \tag{1.10}$$

We formulate the following assumptions:

(H1) $\rho : \bar{\Omega}\times R^2 \to R^2$ satisfies;

(a) $\rho(x,s)s \geq 0,\ s\in R^2,\ x\in\bar{\Omega}$;

(b) ρ and $\dfrac{\partial\rho}{\partial s_i}$ are continuous functions in $\bar{\Omega}\times R^2$;

(c) There exist positive constants $K_0,\ K_1,\ K_2$ and K_3 and numbers $p,\ r,\ -1 < r < \infty,\ -1 < p < \infty$, such that

$$K_2 a(x)|s|^{r+1} \leq |\rho(x,s)| \leq K_0 a(x)(|s|^{r+1} + |s|),\ \text{if } |s|\leq 1,$$

$$K_3 a(x)|s|^{p+1} \leq |\rho(x,s)| \leq K_1 a(x)(|s|^{p+1} + |s|),\ \text{if } |s|\geq 1,$$

(d) $\displaystyle\sum_{i,k=1}^{2} u_i \frac{\partial \rho_i(x,s)}{\partial s_k} u_k \geq 0,\ \forall u \in R^2,\ \forall x\in\bar{\Omega},\ \forall s\in R^2$, i.e., $\frac{\partial\rho}{\partial s}$ is positive semi-definite;

(e) The function $a = a(x)$ in (c) satisfies : $a : \bar{\Omega} \to R^+$ belongs to $L^\infty(\Omega)$.

Remark 1. If $a(x)$ is a continuous function on $\bar{\Omega}$, then $\rho = a(x)|s|^p s,\ s\in R^2$, is an example of a function which satisfies (a)-(e).

2. Existence and regularity

In this section we shall study the existence and regularity of solutions to Eqs. (1.1)-(1.5). To this end we will assume that the relaxation functions g_1 and g_2 are positive and we shall use equations (1.3) and (1.4) to estimate the values $\mathcal{B}_1$ and $\mathcal{B}_2$ on Γ_1. Denoting by

$$(g*\varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating Eqs. (1.3) and (1.4) we arrive at the following Volterra equations:

$$(\mathcal{B}_1 u + \gamma_1\frac{\partial u}{\partial\nu}) + \frac{1}{g_1(0)}g_1' * (\mathcal{B}_1 u + \gamma_1\frac{\partial u}{\partial \nu}) = -\frac{1}{g_1(0)}\frac{\partial u_t}{\partial\nu},$$

$$(\mathcal{B}_2 u - \gamma_2 u) + \frac{1}{g_2(0)}g_2' * (\mathcal{B}_2 u - \gamma_2 u) = \frac{1}{g_2(0)}u_t.$$

Applying the Volterra's inverse operator, we get

$$\mathcal{B}_1 u + \gamma_1 \frac{\partial u}{\partial \nu} = -\frac{1}{g_1(0)}\{\frac{\partial u_t}{\partial \nu} + k_1 * \frac{\partial u_t}{\partial \nu}\},$$
$$\mathcal{B}_2 u - \gamma_2 u = \frac{1}{g_2(0)}\{u_t + k_2 * u_t\},$$

where the resolvent kernels satisfies

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i', \ \forall i = 1, 2.$$

Denoting by $\tau_1 = \dfrac{1}{g_2(0)}$ and $\tau_2 = \dfrac{1}{g_1(0)}$ can be written as

$$\mathcal{B}_1 u = -\gamma_1 \frac{\partial u}{\partial \nu} - \tau_1\{\frac{\partial u_t}{\partial \nu} + k_1(0)\frac{\partial u}{\partial \nu} - k_1(t)\frac{\partial u_0}{\partial \nu} + k_1' * \frac{\partial u}{\partial \nu}\}, \tag{2.1}$$

$$\mathcal{B}_2 u = \gamma_2 u + \tau_2\{u_t + k_2(0)u - k_2(t)u_0 + k_2' * u\}. \tag{2.2}$$

Reciprocally, taking initial data such that $u_0 = 0$ on Γ_1, identity (2.1) and (2.2) imply (1.3) and (1.4). Since we are interested in relaxation function of exponential type and identity (2.1) and (2.2) involve the resolvent kernel k_i, we want to know if k_i has the same properties. The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel, this is

$$k(t) - k * h(t) = h(t). \tag{2.3}$$

Lemma 2.1. ([9]) *If h is a positive continuous function, then k also is a positive continuous function. Moreover,*

1. *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$h(t) \le c_0 e^{-\gamma t},$$

then the function k satisfies

$$k(t) \le \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

Due to this lemma, in the remainder of this paper, we shall use (2.1) and (2.2) instead of (1.3) and (1.4). Let us denote by

$$(g \diamond \varphi)(t) := \int_0^t g(t-s)|\varphi(t) - \varphi(s)|^2 ds.$$

The following lemma state an important property of the convolution operator.

Lemma 2.2. *For $g,\ \varphi \in C^1([0,\infty) : R)$ we have*

$$(g * \varphi)\varphi_t = -\frac{1}{2} g(t)|\varphi(t)|^2 + \frac{1}{2} g' \diamond \varphi - \frac{1}{2}\frac{d}{dt}[g \diamond \varphi - (\int_0^t g(s)ds)|\varphi|^2].$$

The proof of this lemma follows by differentiating the term $g \diamond \varphi$.

Lemma 2.3. *Suppose that* $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_1)$ *and* $h \in H^{3/2}(\Gamma_1)$ *then, any solution of*

$$a(v,w) = \int_\Omega fw dx + \int_{\Gamma_1} gw d\Gamma_1 + \int_{\Gamma_1} h \frac{\partial w}{\partial \nu} d\Gamma_1, \ \forall w \in W$$

satisfies $v \in H^4(\Omega)$ *and also*

$$\Delta^2 v = f, \quad v = \frac{\partial v}{\partial \nu} = 0 \ on \ \Gamma_0, \quad \mathcal{B}_1 v = h, \quad \mathcal{B}_2 v = g \ on \ \Gamma_1.$$

Let us introduce the energy function

$$\begin{aligned} E(t) =& \frac{1}{2}\{\int_\Omega |u_t|^2 dx + a(u,u) + \int_{\Gamma_1} \gamma_1 |\frac{\partial u}{\partial \nu}|^2 + \gamma_2 |u|^2 d\Gamma_1 \\ &+ \tau_2 \int_{\Gamma_1} (k_2(t)|u|^2 - k_2' \diamond u) d\Gamma_1 + \tau_1 \int_{\Gamma_1} (k_1(t) |\frac{\partial u}{\partial \nu}|^2 - k_1' \diamond \frac{\partial u}{\partial \nu}) d\Gamma_1\}, \end{aligned}$$

and let us denote by $\{w_i \in W; i \in N\}$ an orthonormal basis of W. In these conditions we are able to prove the existence of strong solution.

Theorem 2.1. *Let* $k_i \in C^2(R^+)$ *be such that*

$$k_i, \ -k_i', \ k_i'' \geq 0.$$

If $(u_0, \ u_1) \in (H^4(\Omega) \cap W) \times W$ *satisfy the compatibility condition*

$$\mathcal{B}_1 u_0 = \gamma_1 \frac{\partial u_0}{\partial \nu} - \tau_1 \frac{\partial u_1}{\partial \nu}, \tag{2.4}$$

$$\mathcal{B}_2 u_0 = \gamma_2 u_0 + \tau_2 u_1 \quad on \quad \Gamma_1 \tag{2.5}$$

then there is only one solution u *of system (1.1) satisfying*

$$u \in L^\infty(0,T : H^4(\Omega) \cap W), \ u_t \in L^\infty(0,T:W), \ u_{tt} \in L^\infty(0,T:L^2(\Omega)).$$

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