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ENERGY DECAY FOR THE NONLINEAR WAVE EQUATION WITH A HALF-LINEAR DISSIPATION IN $\mathbb{R}^N$

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ABSTRACT. We study decay estimates of the energy for the nonlinear wave equation in the whole space. The dissipative term consists of the following two parts: The first part is nonlinear in suitable ball which contains the obstacle and is effective only in localized area; the second part is linear in the outside of the ball and is effective at infinity. So we may call such a dissipation as half-linear dissipation. We note that the method of proof is based on the multiplier technique and on the unique continuation.

1. INTRODUCTION

In this paper we consider the Cauchy problem for the nonlinear wave equation with a half-linear dissipation;

$$u_{tt} - \triangle u + \rho(x, u_t) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1 \quad \text{in } \mathbb{R}^N \tag{1.2}$$

where $\rho(x, v)$ is some nonlinear function specified later. For the sequel, we need some notations. We set $B_r := \{x \in \mathbb{R}^N ||x| < r\}$ and $\Omega_r := \mathbb{R}^N \setminus B_r$ for $r > 0$.

Let $L > 0$ be arbitrarily fixed and $a(x)$ be a nonnegative bounded function on $\mathbb{R}^N$ such that

$$a(x) \geq \epsilon_0 > 0 \ a.e. \text{ for } x \in \Omega_L. \tag{1.3}$$

We now make the following hypotheses on the dissipative term $\rho(x, v)$.

Hyp.A $\rho(x, v)$ is differentiable *a.e.* and nondecreasing function in v such that

$$\rho(x, v) = \tilde{\rho}(v)\chi(B_L) + a(x)v\chi(\Omega_L), \tag{1.4}$$

where $\tilde{\rho}(v)$ satisfies

$$k_0 a(x)|v|^{r+2} \leq \tilde{\rho}(v)v \leq k_1 a(x)\{|v|^{r+2} + |v|^2\} \text{ for } (x, t) \in B_L \times (0, \infty) \tag{1.5}$$

with $k_0, k_1 > 0$, $0 \leq r \leq 2/(N-2)$, L is given by (1.3), and $\chi(A)$ denotes the characteristic function of A. For an example, $\tilde{\rho}(v)$ is a function like $\tilde{\rho}(v) = a(x)|v|^r v$.

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Condition (1.4) means that the dissipative term $\rho(x, u_t)$ has two character; linear and nonlinear. More precisely, the dissipative term is the linear function $a(x)u_t$ on Ω_L, which is effective at infinity. On the other hand, it is the nonlinear function $\tilde{\rho}(u_t)$ on B_L satisfying (1.5). By reason of such two character, we may call the dissipation the half-linear dissipation and it is meaningful. This half-linear problem was first treated in Nakao and Jung [6]. They obtained the algebraic decay estimates of the energy for the wave equation with some nonlinear dissipative term under two types of parameter in an exterior domain, that is, the decay rates given by algebraic functions as the upper bounds were derived. As a natural question, can we relax the condition of the dissipative term by using any parameter?

The main purpose of this paper is to derive precise decay estimates of the energy for Cauchy problem (1.1)-(1.2) with a nonlinear dissipative term under only one parameter. We note here that since our assumption on the nonlinear dissipative term is given by only one parameter, the upper bounds for our decay rate consist of both functions of algebraic type and functions of logarithmic type depending on the size of the parameter. In this regard, our result extends the results of [6] partially. We remark that our decay result is not a verification of the results in [6].

The problem of proving decay estimates of the solutions to the wave equation with some dissipation has attracted a lot of attention in recent years. To our knowledge, the results in [6] are the only things for the whole space, though the Klein-Gordon type wave equation with nonlinear dissipations like $|u_t|^r u_t$ have been treated by Zuazua[11], Nakao [5], Nakao and Ono [7], Ono [8], and Mochizuki and Motai [4]. Recently, Todorova [10] have analyzed the global existence, nonexistence, and the decay estimates for the Cauchy problem with some perturbed source term of the type :

$$u_{tt} - \triangle u + q^2(x)u + u_t|u_t|^\ell = u|u|^m \quad \text{ in } \mathbb{R}^N \times (0, \infty),$$

where $q(x)$ is a locally bounded measurable function on $\mathbb{R}^N$ and $\ell,\ m \in \mathbb{N}$.

2. Preliminaries and Statement of the Main Result

Throughout this paper we shall use the following notations :

$$||u||_p \equiv ||u||_{L^p(\Omega)},\ 1 \le p < \infty;$$

$H^m(\Omega)(m \ge 0)$ denotes the usual Sobolev space with the norm

$$||f||_{H^m(\Omega)} = \Big(\sum_{|\alpha| \le m} ||D_x^\alpha f(x)||_2^2 dx \Big)^{\frac{1}{2}} < \infty,$$

where α is the multi-indices, and $H_0^m(\Omega)$ is a completion of C_0^∞ in $H^m(\Omega)$ with the above norm. For simplicity, we will write $||u||$ for $||u||_2$.

For the sequel, we assume that

Hyp.B $(u_0, u_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and

$$\text{supp} u_0 \cup \text{supp} u_1 \subset \{x \in \mathbb{R}^N ||x| \le K\} \tag{2.1}$$

for some $K > 0$.

Before stating our main result, let us recall the following well-posedness result, which is given by Lions and Strauss [3].

Theorem 2.1. *Let $(u_0, u_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Then, under Hyp.A and Hyp.B, the problem (1.1)-(1.2) admits a unique solution with the finite propagation property*

$$u(t) \in W^{2,\infty}([0,T); L^2(\mathbb{R}^N)) \cap W^{1,\infty}([0,T); H^1(\mathbb{R}^N)) \cap L^\infty([0,T); H^2(\mathbb{R}^N)$$

for any $T > 0$.

Moreover, for the solution $u(t)$ to the problem (1.1)-(1.2), there exists a finite constant $K > 0$ such that

$$||\nabla u_t|| + ||u_t|| \leq K \text{ for } t \in [0,T). \tag{2.2}$$

The main result of this paper is as follows.

Theorem 2.2. *Let $(u_0, u_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $N \geq 3$. Assume that Hyp.A and Hyp.B are satisfied. Then there exist some positive constant $C_i, i = 1,2,3$ such that the energy $E(t) \equiv 1/2(||u_t||^2 + ||\nabla u||^2)$ for the problem (1.1)-(1.2) satisfy the following decay properties:*

(i) *If $(4(N-1) - \sqrt{8N^2 - 6N})/(N-1)(N-2) < r < 2/N$, then*

$$E(t) \leq C_1(1+t)^{-\gamma}$$

with

$$\gamma = \min\left\{ \frac{(N-1)(N-2)r^2 - 8(N-1)r + 8}{(r+1)(4-(N-2)r)}, \frac{2-Nr}{r}, \frac{3(2-N)r^2 + 2(8-N)r + 8}{4(r+2)} \right\}.$$

(ii) *If $r = 2/N$, then*

$$E(t) \leq C_2(\log(2+t))^{-\frac{2}{r}}.$$

(iii) *If $r = (4(N-1) - \sqrt{8N^2 - 6N})/(N-1)(N-2)$, then*

$$E(t) \leq C_3(\log(2+t))^{-\frac{1}{r+1}}.$$

The proof of Theorem 2.2 relies on the the following lemmas.

First, we need the following well-known lemma without proof here.

Lemma 2.1. *(Gagliardo-Nirenberg) Let $1 \leq r < p \leq \infty$, $1 \leq q \leq p$. Then we have the inequality*

$$||v||_{H^p(\mathbb{R}^N)} \leq c||v||^{\theta}_{H^q(\mathbb{R}^N)}||v||_r^{1-\theta} \text{ for } v \in H^p(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$$

with some $c > 0$ and

$$\theta = \left(\frac{2}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{2}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \theta \leq 1$ ($0 < \theta < 1$ if $p = \infty$ and $2q = N$).

Lemma 2.2. ([5]) *Let $\phi(t)$ be a nonnegative function on $[0,\infty)$ satisfying the inequality*

$$\sup_{t\leq s\leq t+T}\phi(s)\leq C\sum_{i=1}^{2}(1+t)^{\theta_i}(\phi(t)-\phi(t+1))^{\epsilon_i},\ t\geq 0$$

with some $T>0$, $C>0$, $0<\epsilon_i\leq 1$ and $\theta_i\leq\epsilon_i$. Then $\phi(t)$ has the following decay property:

(1) *If $0<\epsilon_i\leq 1$ and $\theta_i<\epsilon_i, i=1,2$, then*

$$\phi(t)\leq C_0(1+t)^{-\alpha}$$

with $\alpha=\min_{i=1,2}\{(\epsilon_i-\theta_i)/(1-\epsilon_i)\}$.

(2) *If $\theta_1=\epsilon_1<1$ and $\theta_2<\epsilon_2\leq 1$, then*

$$\phi(t)\leq C_0(\log(2+t))^{-\frac{\epsilon_1}{1-\epsilon_1}}.$$

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