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ON A CLASS OF GENERALIZED LOGARITHMIC FUNCTIONAL EQUATIONS IN ALMOST EVERYWHERE SENSE

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ABSTRACT. Making use of the theory of Schwartz distributions we consider a class of generalized logarithmic functional equations in almost everywhere sense.

1. INTRODUCTION

Considering a functional equation whose underlining functions are defined on a measure space it is more natural to deal with the equation in *almost everywhere sense* than *for all sense*. Recently, some functional equations have been studied in the sense of Schwartz distributions which can be regarded as such an approach [1, 2, 3, 4, 5]. In this article we consider the following Pexiderized logarithmic functional equations in almost everywhere sense

$$f(x+y) - g(xy) = h(1/x + 1/y), \quad a.e.\ x, y > 0, \tag{1.1}$$

$$f(x+y) - g(x) - h(y) = k(1/x + 1/y), \quad a.e.\ x, y > 0, \tag{1.2}$$

$$f\left(\frac{x+y}{2}\right) + g\left(\frac{2xy}{x+y}\right) = h(x) + k(y), \quad a.e.\ x, y \in I, \tag{1.3}$$

where $I \subset (0, \infty)$ is an open interval. As results, reducing the equation (1.1), (1.2) and (1.3) to differential equations, which is one of the most powerful advantages of the Schwartz theory, we find the locally integrable solutions of the equation (1.1), (1.2) and (1.3). We refer the reader to [1, 4, 5] for more results using this method of reducing given functional equations to differential equations.

2. DISTRIBUTIONAL APPROACH TO GENERALIZED LOGARITHMIC FUNCTIONAL EQUATIONS

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{D}'(\Omega)$ the space of Schwartz distributions on Ω. Recall that a distribution u is a linear functional on $C_c^\infty(\Omega)$ of infinitely differentiable functions on Ω with compact supports such that for every compact set $K \subset \Omega$ there exist constants C and k satisfying

$$|\langle u, \varphi \rangle| \le C \sum_{|\alpha| \le k} \sup |\partial^\alpha \varphi|$$

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for all $\varphi \in C_c^\infty(\Omega)$ with supports contained in K. Here we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We briefly introduce some basic operations in $\mathcal{D}'(\Omega)$. Let $u \in \mathcal{D}'(\Omega)$. Then the k-th partial derivative $\partial_k u$ of u is defined by

$$\langle \partial_k u, \, \phi \rangle = -\langle u, \, \partial_k \phi \rangle$$

for $k = 1, \ldots, n$. Let $f \in C^\infty(\Omega)$. Then the multiplication fu is defined by

$$\langle fu, \, \phi \rangle = \langle u, \, f\phi \rangle.$$

We denote by Ω_j open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$.

Definition 2.1. *Let $u_j \in \mathcal{D}'(\Omega_j)$ and $f : \Omega_1 \to \Omega_2$ a smooth function such that for each $x \in \Omega_1$ the derivative $f'(x)$ is surjective. Then there exist a unique continuous linear map $f^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $f^*u = u \circ f$ when u is a continuous function. We call f^*u the pullback of u by f and often denoted by $u \circ f$.*

Every locally integrable function $f : (0, \infty) \to \mathbb{C}$ can be viewed as a distribution via the equation

$$\langle f, \varphi \rangle = \int f(x)\varphi(x) \, dx,$$

for all $\varphi \in C_c^\infty((0, \infty))$. Consequently the equations (1.1), (1.2) can be viewed in the sense of functional on $C_c^\infty((0, \infty)^2)$ and the equation (1.3) can be viewed in the sense of functional on $C_c^\infty(I^2)$. By virtue of the Schwartz theory we can differentiate the locally integrable functions freely in the sense of distributions and reduce the equations to differential equations which is very useful method of finding the solutions of the logarithmic functional equations.

From the above definitions we can check that the usual Leibniz rule and chain rule for differentiations are fulfilled in the space of distributions.

Theorem 2.2. *Every locally integrable solution f, g, h of the equation* (1.1) *has the form*

$$f(z) = c_1 + c_2 + a \ln z, \;\; a.e. \; z > 0 \tag{2.1}$$

$$g(z) = c_1 + a \ln z, \;\; a.e. \; z > 0 \tag{2.2}$$

$$h(z) = c_2 + a \ln z, \;\; a.e. \; z > 0 \tag{2.3}$$

where $a, c_1, c_2 \in \mathbb{C}$.

Proof. We denote by P, R, P_1, P_2 the functions $P(x, y) = xy, R(x, y) = 1/x + 1/y, P_1(x, y) = x, P_2(x, y) = y$. Applying $\partial_1 - \partial_2$ in (1.1) we have

$$(x - y)(g' \circ P) = \frac{x^2 - y^2}{x^2 y^2} \, (h' \circ R),$$

which implies

$$g' \circ P = \frac{x + y}{x^2 y^2} \, (h' \circ R), \tag{2.4}$$

on the space $C_c^\infty(V)$, where $V = \{(x, y) : x > y > 0\}$. Let $J : V \to U := \{s, t) : st^2 > 4, t > 0\}$ be a diffeomorphism defined by $J(x, y) = (xy, 1/x + 1/y)$. Then, since $(P \circ J^{-1})(s, t) = s$, $(R \circ J^{-1})(s, t) = t$, taking pullback of (2.4) by J^{-1}, we have

$$(zg'(z)) \circ P_1 = (zh'(z)) \circ P_2 \tag{2.5}$$

as distributions in $\mathcal{D}'(U)$, where $P_1(s, t) = s$, $P_2(s, t) = t$. Applying ∂_1 in (2.5) (or localize and apply tensor product of test functions) it follows that

$$zg'(z) = zh'(z) := a \tag{2.6}$$

are the same constants. Since the solutions $g, h \in \mathcal{D}'((0, \infty))$ of the equation (2.6) are the same as the classical solutions of the equation we get (2.2) and (2.3). Choose $\psi \in C_c^\infty((0, \infty))$ such that $\int \psi(y)dy = 1$. For given $\varphi \in C_c^\infty((0, \infty))$, applying $\phi(x, y) = \varphi(x + y)\psi(y)$ in (1.1) and using (2.2) and (2.3), it follows that

$$\langle f, \varphi\rangle = \int (c_1 + c_2 + a \ln z)\varphi(z)dz,$$

for all $\varphi \in C_c^\infty((0, \infty))$, which gives (2.1). This completes the proof. □

As a consequence of the above result we have the following.

Corollary 2.3. *Every locally integrable solution f, g, $h : (0, \infty) \to \mathbb{C}$ of the equation*

$$f(x + y) - g(xy) = h(1/x + 1/y), \ x, y > 0,$$

has the form

$$f(z) = c_1 + c_2 + a \ln z, \tag{2.7}$$

$$g(z) = c_1 + a \ln z, \tag{2.8}$$

$$h(z) = c_2 + a \ln z, \tag{2.9}$$

where $c_1, c_2, a \in \mathbb{C}$.

Proof. It follows from Theorem 2.2 that the equations (2.7), (2.8) and (2.9) hold for all z in a subset $\Omega \subset (0, \infty)$ with $m(\Omega^c) = 0$. For given $z > 0$, let p, $q : (0, \infty) \to \mathbb{R}$ by $p(t) = t + z/t$, $q(t) = 1/t + t/z$. Then since $m(p^{-1}(\Omega)^c \cup q^{-1}(\Omega)^c) = 0$, we have $p^{-1}(\Omega) \cap q^{-1}(\Omega) \neq \emptyset$. Thus we can choose x, $y > 0$ so that $xy = z$ and $x + y$, $1/x + 1/y \in \Omega$. It follows from (1.1), (2.7) and (2.9) that

$$\begin{aligned} g(z) &= f(x + y) - h(1/x + 1/y) \\ &= c_1 + a \ln(xy) = c_1 + a \ln z, \end{aligned} \tag{2.10}$$

which means that the equation (2.8) holds for all $z > 0$. Similarly, let $p : (0, z) \to \mathbb{R}$ by $p(t) = 1/t + 1/(z - t)$. Then we have $p^{-1}(\Omega) \neq \emptyset$. Thus we can choose x, $y > 0$ so that $x + y = z$ and $1/x + 1/y \in \Omega$. Thus it follows from (1.1), (2.7) and (2.8) that the equation(2.9) holds for all $z > 0$, and then, from (1.1) the equation (2.7) also holds for all $z > 0$. This completes the proof. □

Theorem 2.4. *Every locally integrable solution f, g, h, k of the equation* (1.2) *has the form*

$$f(z) = -a\ln z + bz + c_1,\ \ a.e.\ z > 0 \tag{2.11}$$

$$g(z) = -a\ln z + bz - d/z + c_1 + c_3,\ \ a.e.\ z > 0 \tag{2.12}$$

$$h(z) = -a\ln z + bz - d/z - c_2 - c_3,\ \ a.e.\ z > 0 \tag{2.13}$$

$$k(z) = -a\ln z + dz + c_2,\ \ a.e.\ z > 0 \tag{2.14}$$

where $a, b, d, c_1, c_2, c_3 \in \mathbb{C}$.

Proof. Applying $\partial_1\partial_2$ in (1.2) we have

$$f'' \circ S = \frac{1}{x^2y^2}\,(k'' \circ R), \tag{2.15}$$

where $S(x, y) = x + y$. Similarly as in the proof of Theorem 2.2, define a diffeomorphism $J : V = \{x, y) : x > y > 0\} \to U := \{s, t) : s > 0,\ st > 4\}$ by $J(x, y) = (x + y, 1/x + 1/y)$, and taking pullback of (2.15) by J^{-1} we have

$$(z^2 f''(z)) \circ P_1 = (z^2 k''(z)) \circ P_2 \tag{2.16}$$

as distributions in $\mathcal{D}(U)$. Applying ∂_1 in (2.16) it follows that

$$z^2 f''(z) = z^2 k''(z) := a \tag{2.17}$$

are the same constant. The solution $f, k \in \mathcal{D}'((0,\infty))$ of the equation (2.17) are given by (2.11) and (2.14). Put (2.11) and (2.14) in (1.2) to get

$$\begin{aligned} g \circ P_1 + h \circ P_2 =& f \circ S - k \circ R \\ =& -a\ln(x + y) + b(x + y) + c_1 \\ & + a\ln(1/x + 1/y) - d(1/x + 1/y) - c_2 \\ =& -(a\ln x - bx + d/x - c_1) \\ & -(a\ln y - by + d/y + c_2). \end{aligned} \tag{2.18}$$

Thus it follows that

$$(g + a\ln z - bz + d/z - c_1) \circ P_1 = -(h + a\ln z - bz + d/z + c_2) \circ P_2,$$

and hence

$$g + a\ln z - bz + d/z - c_1 = -(h + a\ln z - bz + d/z + c_2) := c_3$$

are the same constants, which gives (2.12) and (2.13). This completes the proof. □

As a consequence of the Theorem 2.3 we have the following.

Corollary 2.5. *Every locally integrable solution f, g, h, k : $(0,\infty) \to \mathbb{C}$ of the equation*

$$f(x + y) - g(x) - h(y) = k(1/x + 1/y)$$

has the form

$$f(z) = -a \ln z + bz + c_1, \tag{2.19}$$

$$g(z) = -a \ln z + bz - d/z + c_1 + c_3, \tag{2.20}$$

$$h(z) = -a \ln z + bz - d/z - c_2 - c_3, \tag{2.21}$$

$$k(z) = -a \ln z + dz + c_2, \tag{2.22}$$

where $a, b, d, c_1, c_2, c_3 \in \mathbb{R}$.

Proof. From the above result the equation (2.19) $\sim$ (2.22) hold for all z in a subset $\Omega \subset (0, \infty)$ with $m(\Omega^c) = 0$. For given $x > 0$, let p, $q : (0, \infty) \to \mathbb{R}$ by $p(t) = x + t$, $q(t) = 1/x + 1/t$. Then since $\Omega \cap p^{-1}(\Omega) \cap q^{-1}(\Omega) \neq \emptyset$, we can choose $y > 0$ so that y, $x + y$, $1/x + 1/y \in \Omega$. It follows from (1.2), (2.19), (2.21) and (2.22) that

$$\begin{aligned} g(x) =& f(x + y) - h(y) - k(1/x + 1/y) \\ =& - a \ln(x + y) + b(x + y) + c_1 \\ &+ a \ln y - by + d/y + c_2 + c_3 \\ &+ a \ln(1/x + 1/y) - d(1/x + 1/y) - c_2 \\ =& - a \ln x + bx - d/x + c_1 + c_3, \end{aligned}$$

which means that the equation (2.20) holds for all $z > 0$. Exchanging x and y in (1.2) and following the sane approach as above we get the equality (2.21). Similarly, for given $z > 0$, we can choose x, $y \in \Omega$ so that $x + y = z$, $1/x + 1/y \in \Omega$ and the equation (2.19) follows from (1.2), (2.20), (2.21) and (2.22). Finally, the equation (2.22) follows from (1.2) and the equalities (2.19), (2.20) and (2.21). □

We denote by M, H the functions $M(x, y) = (x + y)/2$, $H(x, y) = 2xy/(x + y)$.

Theorem 2.6. *Every locally integrable solution* f, g, h, k *of the equation* (1.3) *has the form*

$$f(z) = az + b \ln z + c_1, \text{ a.e. } z \in I \tag{2.23}$$

$$g(z) = b \ln z - 2d/z + c_2 + c_3 - c_1, \text{ a.e. } z \in I \tag{2.24}$$

$$h(z) = \frac{az}{2} + b \ln z - d/z + c_2, \text{ a.e. } z \in I \tag{2.25}$$

$$k(z) = \frac{az}{2} + b \ln z - d/z + c_3, \text{ a.e. } z \in I \tag{2.26}$$

where $a, b, d, c_1, c_2, c_3 \in \mathbb{C}$.

Proof. Applying $x^2\partial_1 - y^2\partial_2$ in (1.3) we have

$$\frac{1}{2}(x^2 - y^2)(f' \circ M) = x^2(h' \circ P_1) - y^2(k' \circ P_2). \tag{2.27}$$

If we denote by f^*, h^*, k^*,

$$f^*(z) = zf'(z), \; h^*(z) = z^2h'(z), \; k^*(z) = z^2k'(z), \tag{2.28}$$

the equation (2.27) is written

$$(x - y)(f^* \circ M) = h^* \circ P_1 - k^* \circ P_2. \tag{2.29}$$

Applying $\partial_1 - \partial_2$ in (2.29) we have

$$2(f^* \circ M) = (h^*)' \circ P_1 + (k^*)' \circ P_2. \tag{2.30}$$

As a local version of the Jensen-Pexider type functional equation[2, Theorem 3.4], it is easy to see that the solution f^*, $(h^*)'$, $(k^*)'$ has the form

$$f^*(z) = az + (b+c)/2,\ (h^*)'(z) = az + b,\ (k^*)'(z) = az + c, \tag{2.31}$$

and it follows that

$$f^*(z) = az + (b+c)/2,\ h^*(z) = \frac{a}{2}z^2 + bz + d_1,\ k^*(z) = \frac{a}{2}z^2 + cz + d_2. \tag{2.32}$$

Put (2.32) in (2.29) to get the equality

$$\left(\frac{b+c}{2}\right)(x-y) = bx - cy + d_1 - d_2$$

as distributions in $\mathcal{D}'(I^2)$, which implies $b = c$, $d_1 = d_2 := d$. Thus we have

$$f^*(z) = az + b,\ h^*(z) = k^*(z) = \frac{a}{2}z^2 + bz + d. \tag{2.33}$$

In view of (2.28) and (2.33) we have (2.23), (2.25) and (2.26). Put (2.23), (2.25) and (2.26) in (1.3) to get (2.24). This completes the proof.

□

Similarly as in the proof of Corollary 2.3 and Corollary 2.5 we have the followings.

Corollary 2.7. *Every locally integrable solution f, g, h, $k : I \to \mathbb{C}$ of the equation*

$$f\left(\frac{x+y}{2}\right) + g\left(\frac{2xy}{x+y}\right) = h(x) + k(y),\ x, y \in I$$

has the form

$$f(z) = az + b\ln z + c_1, \tag{2.34}$$

$$g(z) = b\ln z - 2d/z + c_2 + c_3 - c_1, \tag{2.35}$$

$$h(z) = \frac{az}{2} + b\ln z - d/z + c_2, \tag{2.36}$$

$$k(z) = \frac{az}{2} + b\ln z - d/z + c_3, \tag{2.37}$$

where $a, b, d, c_1, c_2, c_3 \in \mathbb{C}$.

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