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STABILIZATIONS OF HEEGAARD SPLITTINGS AND UNKNOTTING TUNNELS

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ABSTRACT. In this article, we consider questions about stabilizations of Heegaard splittings and specially for the stabilizations of splittings induced by unknotting tunnels.

1. INTRODUCTION

A *compression body* H is a 3-manifold obtained from a compact connected closed surface S by attaching 2-handles to $S \times I$ on $S \times \{1\}$ and capping off any resulting 2-sphere boundary components with 3-handles. $S \times \{0\}$ is denoted by $\partial_+ H$ and $\partial H - \partial_+ H$ is denoted by $\partial_- H$. A compression body H is called a *handlebody* if $\partial_- H = \emptyset$.

If a compact 3-manifold M is the union of two compression bodies H_1 and H_2 along their common "plus" boundary $S = \partial_+ H_1 = \partial_+ H_2$, we call the decomposition $M = H_1 \cup_S H_2$ a *Heegaard splitting* of M and S a *Heegaard surface* of M. The minimum number of the genus of S among all Heegaard splittings of M is called the *Heegaard genus* (or *genus*) of M.

Suppose $H_1 \cup_S H_2$ is a Heegaard splitting of a 3-manifold M and α is a properly embedded arc in H_2 parallel to an arc in S. That is, there is an embedded disk D in H_2 whose boundary is the union of α and an arc in $\partial_+ H_2$. Now add a neighborhood of α to H_1 and delete it from H_2. Once again the result is a Heegaard splitting $H_1' \cup_{S'} H_2'$, where the genus of each H_i' is one greater than H_i. This process is called a *stabilization* of S.

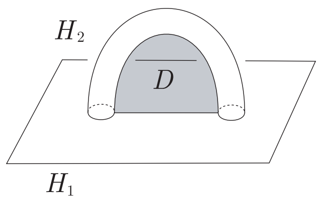

FIGURE 1. stabilization

2. CONNECTED SUM AND STABILIZATION

There was an open problem on the connected sum of Heegaard splittings.

Conjecture 2.1. *(Gordon) The connected sum of two Heegaard splittings is stabilized if and only if one of the two Heegaard splittings is stabilized.*

Recently this conjecture was proved by Qiu.

Theorem 2.2. [8] *The connected sum of two Heegaard splittings is stabilized if and only if one of the two Heegaard splittings is stabilized.*

3. Stabilization problem

Every compact 3-manifold can be triangulated and any two triangulations of a 3-manifold are PL-equivalent ([1], [6]). It follows that every compact 3-manifold has a Heegaard splitting and any two Heegaard splittings of a 3-manifold have a common stabilization. In fact, there is no example of distinct Heegaard splittings of a same closed 3-manifold which cannot be made isotopic by a single stabilization of one of the splittings, and sufficient stabilizations of the other to ensure that the genus of the two surfaces is the same. This makes the following conjecture very optimistic.

Conjecture 3.1. [10] *Suppose $H_1 \cup_S H_2$ and $H_1^{'} \cup_{S'} H_2^{'}$ are Heegaard splittings of the same 3-manifold of, genus $g \leq g'$ respectively. Then the splittings obtained by one stabilization of S' and $g' - g + 1$ stabilizations of S are isotopic.*

On the other hand, there are results which put limits on how much stabilizations are needed, in terms of the genera of the two original splittings. Here a Heegaard splitting $H_1 \cup_S H_2$ is called *strongly irreducible* if for any essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$, D_1 and D_2 intersect.

Theorem 3.2. [9] *Suppose $X \cup_Q Y$ and $A \cup_P B$ are strongly irreducible Heegaard splittings of the same closed orientable 3-manifold M and are of genus $p \leq q$ respectively. Then there is a genus $8q + 5p - 9$ Heegaard splitting of M which stabilize both $A \cup_P B$ and $X \cup_Q Y$.*

The proof is an application of Heegaard splittings as sweep-outs and Rubinstein-Scharlemann graphic.

4. Unknotting tunnel and stabilization

A *tunnel system* (or *tunnels*) of a knot or a link K is a collection of disjoint embedded arcs $t_1, t_2, \cdots, t_n$ in S^3 with $K \cap \overset{n}{\underset{i=1}{\cup}} t_i = \overset{n}{\underset{i=1}{\cup}} \partial t_i$ such that $H = \overline{S^3 - N(K \cup \overset{n}{\underset{i=1}{\cup}} t_i)}$ is a genus $n+1$ handlebody. (Here $N(X)$ denotes a regular neighborhood of X.) The tunnel system gives rise to a Heegaard splitting of the exterior of K

$$\overline{S^3 - N(K)} = H \cup_{\partial H} \overline{N(K \cup \overset{n}{\underset{i=1}{\cup}} t_i) - N(K)}$$

where $N(K)$ is contained in the interior of $N(K \cup \overset{n}{\underset{i=1}{\cup}} t_i)$. The minimum of such number n is called the *tunnel number* of K. If the tunnel number of K is 1, the tunnel is called an *unknotting tunnel* of K.

For a tunnel number one knot K, we consider two non-isotopic unknotting tunnels t_1, t_2 and corresponding Heegaard surfaces S_1, S_2 of the exterior of K. Now suppose $V = \overline{S^3 - N(K \cup t_1 \cup t_2)}$ is a genus three handlebody. This means that $S' = \partial V$ becomes a Heegaard surface for the genus two handlebodies $\overline{S^3 - N(K \cup t_1)}$ and $\overline{S^3 - N(K \cup t_2)}$. By [11], there is at most one Heegaard splitting of a handlebody of a given genus. This implies S' is a common stabilization of S_1 and S_2 and shows a validity of Conjecture 3.1. There are examples of knots having this property — torus knots and 2-bridge knots.

4.1. **Torus knots.** A *torus knot* is a knot on the standard torus embedded in S^3. A torus knot can be characterized by two relatively prime integers p and q. $K_{p,q}$ is a torus knot that winds the standard torus p times in meridional direction and q times in longitudinal direction. A torus knot has 3 types of unknotting tunnels t_p, t_q, t_0 (Figure 2) and they are classified in [2].

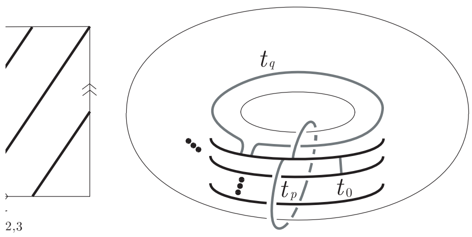

FIGURE 2. $K_{2,3}$ and unknotting tunnels of a torus knot

Theorem 4.1. [2]*(Boileau-Rost-Zieschang) Let $K_{p,q}$ be a torus knot of type (p,q), where* $\gcd(p,q) = 1$ *and* $p > q > 1$.

(1) *Any unknotting tunnel of $K_{p,q}$ is isotopic to t_p, t_q or t_0.*
(2) *t_0 is isotopic to t_p* $\iff q \equiv \pm 1 \bmod p$.
(3) *t_0 is isotopic to t_q* $\iff p \equiv \pm 1 \bmod q$.
(4) *t_p is isotopic to t_q* $\iff |p - q| = 1$.

In [2], they also proved that the two Heegaard splittings given by any two unknotting tunnels among t_p, t_q, t_0 of a torus knot, say t_p and t_q, have a common stabilization $S' = \partial N(K \cup t_p \cup t_q)$.

4.2. **2-bridge knots.** S^3 can be understood as a gluing of two 3-balls along the boundary spheres. A 2-*bridge knot* is a knot which can be decomposed into two trivial 2-string tangles in those two 3-balls. A 2-bridge knot has 6 types of unknotting tunnels s_1, s_1', t_1, s_2, s_2', t_2 (Figure 3) and they are classified in [7].

In [3], Hagiwara showed that the two Heegaard splittings given by any two unknotting tunnels among t_1, s_1, s_1', t_2, s_2, s_2' of a 2-bridge knot have a common stabilization.

When we have two disjoint unknotting tunnels t_1, t_2 of a knot K, $\partial N(K \cup t_1 \cup t_2)$ may not be a Heegaard surface even if t_1 and t_2 are isotopic tunnels. Take t_2 as a

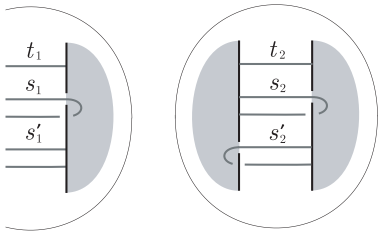

FIGURE 3. Unknotting tunnels of a 2-bridge knot

parallel copy of t_1. Pull a part of t_2 in a complicated way and hook it to t_1. This construction does not give a genus three Heegaard surface (Figure 4). So there must be some restrictions on the choice of the unknotting tunnels.

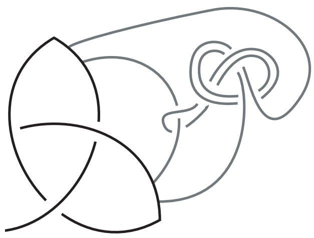

FIGURE 4. Embedding of tunnels which does not give a genus three splitting

In [5] we gave a sufficient condition for a non-trivial tunnel number one link exterior to have a common stabilization when we have two disjoint unknotting tunnels.

Theorem 4.2. [5] *Let $L = K_1 \cup K_2$ be a non-trivial tunnel number one link and t_1 and t_2 be two disjoint unknotting tunnels of L such that a meridian disk D of the genus two handlebody $V = \overline{S^3 - N(L \cup t_1)}$ does not intersect t_2. Then $\overline{S^3 - N(L \cup t_1 \cup t_2)}$ is a genus three handlebody.*

4.3. **2-bridge links.** Let $S_i = s_i \cup s_i'$ be a trivial 2-string tangle in a 3-ball $B_i (i = 1, 2)$. Gluing B_1 and B_2 along their boundary spheres so that $s_1 \cup s_2$ and $s_1' \cup s_2'$ are simple closed curves, we obtain a 2-bridge link $L = S_1 \cup S_2$. We assume that L is non-trivial. By [4], there are two types of unknotting tunnels for 2-bridge links. If two unknotting tunnels of L are parallel, we can easily find a stabilizing disk. So assume that the two tunnels are in standard positions t_1 and t_2 as in Figure 5.

Note that $\overline{B_2 - N(S_2)}$ is a genus two handlebody, and $\overline{B_1 - N(S_1 \cup t_1)}$ is homeomorphic to $(\overline{B_1 - N(S_1 \cup t_1)} \cap \partial B_1) \times I$. Let $f : \partial B_1 \to \partial B_2$ be the gluing homeomorphism.

Let D_1 be the meridian disk as in Figure 5 and $a = \partial D_1 \cap \partial B_2$. Let $b = f(a)$ and D_2 be a disk in $\overline{B_1 - N(S_1 \cup t_1)}$ corresponding to $b \times I \subset (\overline{B_1 - N(S_1 \cup t_1)} \cap \partial B_1) \times I$. Then $D = D_1 \cup_f D_2$ is a meridian disk of genus two handlebody

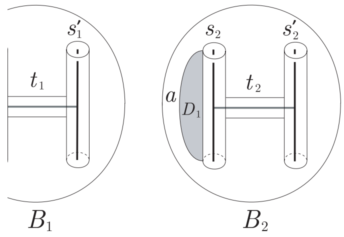

FIGURE 5. Unknotting tunnels of a 2-bridge link

$\overline{S^3 - N(S_1 \cup S_2 \cup t_1)}$ that does not intersect t_2. So this is an example that satisfies the hypothesis of Theorem 4.2 and t_1 and t_2 give a common stabilization.

In Theorem 4.2 when $D \cap t_2 \neq \emptyset$, it is difficult to apply same arguments as in [5]. But recently we got a result with more assumptions when $|D \cap t_2| = 1$.

Proposition 4.3. *Let $L = K_1 \cup K_2$ be a non-trivial tunnel number one link and t_1 and t_2 be two disjoint unknotting tunnels of L. Let D_1 and D_2 be two non-separating, non-parallel meridian disks of $\overline{S^3 - N(L \cup t_1)}$ and suppose $|D_1 \cap t_2| = |D_2 \cap t_2| = 1$. Then $\overline{S^3 - N(L \cup t_1 \cup t_2)}$ is a genus three handlebody.*

We can find an example satisfying the assumptions of Proposition 4.3, for example, a Hopf link.

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