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DAMPED NEWTON'S METHOD FOR IMAGE RESTORATION

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1. Preparation

As in image enhancement, the ultimate goal of restoration techniques is to improve an image in some sense. For the purpose of differentiation, we consider restoration to be a process that attempts to reconstruct or recover an image that has been degraded by using some a priori knowledge of the degradation phenomenon. Thus restoration techniques are oriented toward modeling the image. This approach usually involves formulating a criterion of goodness that will yield some optimal estimate of the desired result. By contrast, enhancement techniques basically are heuristic procedures designed to manipulate an image in order to take advantage of the psychophysical aspects it might present to the viewer, whereas removal of image blur by applying a deblurring function is considered a restoration technique([6], [9]).

The problem of denoising, or estimating an underlying function from error contaminated observations, occurs in a number of important applications, particularly in probability density estimation and image reconstruction. Consider the model equation

$$u_0 = u + n \tag{1.1}$$

where u represents the desired true solution, n represents error, and u_0 represents the observed data. A number of approaches can be taken to estimate u. These include selective smoothing (see [1]), filtering using Fourier and wavelet transforms, level set method, and Total variation (TV) based denoising. Suffice it to say that TV denoising is extremely effective for recovering "block", possibly discontinuous, functions from noise data. It is the goal of this paper to present a new algorithm for TV denoising and introduce some existing TV-based methods.

From 1.1 we consider the minimization of the TV-penalized least squares functional

$$f(u) = \alpha \int_\Omega |\nabla u|_\beta + \frac{1}{2}\|u - u_0\|^2 \tag{1.2}$$

where

$$\int_\Omega |\nabla u|_\beta = \int_\Omega \sqrt{|\nabla u|^2 + \beta^2}$$

and α and β are positive parameters.

Assuming homogeneous Neumann boundary conditions, the functional can be written as

$$0 = g(u) = -\alpha\nabla\cdot\left(\frac{\nabla u}{|\nabla u|_\beta}\right) + u - u_0. \tag{1.3}$$

A number of methods have been proposed to solve 1.3. Rudin, et al,[2] used a time marching scheme to reach the steady state of the parabolic equation $u_t = -g(u)$ with initial condition $u = u_0$. This method can be slowly convergent due to stability constraints. C. Vogel and M. Oman [3] proposed the following fixed point iteration to solve the equation 1.3:

$$0 = -\alpha\nabla\cdot\left(\frac{\nabla u^{k+1}}{|\nabla u^k|_\beta}\right) + u^{k+1} - u_0.$$

At each step, a linear differential convolution equation has to be solved.

One of the difficulties in solving the equation 1.3 is the presence of a highly nonlinear and non-differentiable term, which cause convergence difficulties for Newton's method even when combined with a globalization technique such as a line search. The idea of Chan, et al,[4] is to remove some of the singularity caused by the non-differentiability of the objective function before we apply a linearization technique such as Newton's method. This is accomplished by introducing an additional variable for the flux quantity appearing in the gradient of the objective function, which can be interpreted as the unit normal to the level sets of the image function.

The idea of the method [4] is to introduce

$$w = \frac{\nabla u}{|\nabla u|}$$

as a new variable and replace 1.3 by following equivalent system of nonlinear partial differential equations:

$$\begin{aligned} |\nabla u|w - \nabla u &= 0 \\ -\alpha\nabla\cdot w + u - u_0 &= 0, \end{aligned}$$

where $|\cdot| = |\cdot|_\beta$. They can then linearize this (u, w) system, for example by Newton's method. This approach is similar to the technique of introducing a flux variable in the mixed finite element method.

2. Damped Newton's method

Applying Newton's Method on problem 1.3 has some problems as we know generally. To overcome this problems we introduce a damped Newton method([5], [7], [8]).

Let us first consider a mixed boundary value problem of the form

$$\begin{cases} -\alpha\nabla\cdot\left(\dfrac{\nabla u}{|\nabla u|}\right)+u= & u_0 \qquad \text{in } \Omega \\ \dfrac{\nabla u}{|\nabla u|}\cdot\nu=0 & \text{on } \Gamma_N \\ u=0 & \text{on } \Gamma_D \end{cases}$$

where ν is outward normal vector. Let us regard Ω as a rectangle, because shape of almost all of image is a rectangle.

The Sovolev space

$$H_D^1(\Omega):=\{u\in H^1(\Omega): u_{|\Gamma_D}=0\}$$

defined in corresponding to the Dirichlet boundary Γ_D.

Put

$$F(u)=-\alpha\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)$$

and

$$f(x,\nabla u)=\alpha\left(\frac{\nabla u}{|\nabla u|}\right).$$

Construction *Let $u_0\in H_D^1(\Omega)$ be arbitrary, and let the sequences (u_n) be defined by the following iteration. If, for $n\in N$, u_n is obtained, then*

$$u_{n+1}=u_n+\tau_n p_n \tag{2.1}$$

where $p_n\in H_D^1(\Omega)$ is the solution of the linear auxiliary problem

$$\langle (F'(u_n)+1)p_n, v\rangle_{H_D^1}=-\langle F(u_n)+u-u_0, v\rangle_{H_D^1} \qquad (v\in H_D^1(\Omega))$$

or

$$\int_\Omega\left(\frac{\partial f}{\partial\eta}(x,\nabla u_n)\nabla p_n\cdot\nabla v+p_n v\right)=-\int_\Omega\left(f(x,\nabla u_n)\cdot\nabla v+(u-u_0)v\right), \tag{2.2}$$

further,

$$\tau_n=\min\left\{1,\frac{\mu_1}{L\|p_n\|_{H_D^1}}\right\}\in(0,1]$$

where L and u_1 are constants.

In order to prove convergence of the sequence $\{u_n\}$ from 2.1 we suppose the following assumptions and proved conditions regarding convergence:

Assumptions

a) $\Omega\subset\mathbb{R}^n$ is a bounded domain with piecewise smooth boundary, $\Gamma_N,\Gamma_D\subset\partial\Omega$ are measurable, $\Gamma_N\cap\Gamma_D=\emptyset, \Gamma_N\cup\Gamma_D=\partial\Omega$ and $\Gamma_D\neq\emptyset$;
b) $u_0\in L^2(\Omega)$;

Conditions

i) The function $f:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is measurable and bounded w.r. to the variable

$x \in \Omega$ and C^1 w.r. to variable $\eta \in \mathbb{R}^n$;
ii) The Jacobians $\partial f(x, \eta)/\partial \eta$ are symmetric and their eigenvalues λ satisfy

$$0 < \mu_1 \leq \lambda \leq \mu_2 < \infty$$

with constants $0 < \mu_1 \leq \mu_2 <$ independent of (x, η);

Consequence
From assumptions and conditions, the equation

$$F(u) + u = u_0$$

has a unique solution $u^* \in H_D^1(\Omega)$ *and the sequence* $\{u_n\}$ *converges to the solution* u^* *satisfying the inequality*

$$\|u_n - u^*\|_{H_D^1} \leq \mu_1^{-1} \|F(u_n) + u_n - u_0\|_{H_D^1} \to 0$$

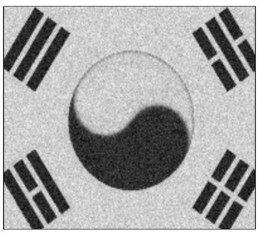
with speed of locally quadratic order. (see [5])

3. Numerical implementation and conclusion

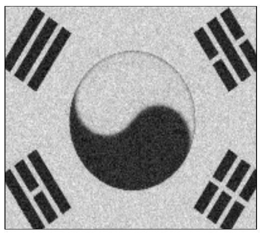
In this paper image is in 256 gray levels and the observed image u_0 is made by adding noise with standard deviation (σ_n). The noise n is assumed to be a *Gaussian white noise*, i.e. , the values $n(x, y)$ are uncorrelated random variables with a normal distribution with mean 0 and variance σ_n^2, for all $(x, y) \in \Omega$.

The images u (resp. u_0) are assumed to be discretized in $m = m_h \times m_v$ (m_h, horizontal size, m_v vertical size) square pixels of dimension $h \times h$ which we order row wise in a vector y (resp. y_0). We use FEM (Finite Elements Method) for 2.2 with boundary conditions and obtain p_n. And then, by using the discrete and equation 2.1 we can obtain next solution.

To produce the following results with pictures file which horizontal size is 200 pixels and vertical size is 160 pixels, I replace α into 5.

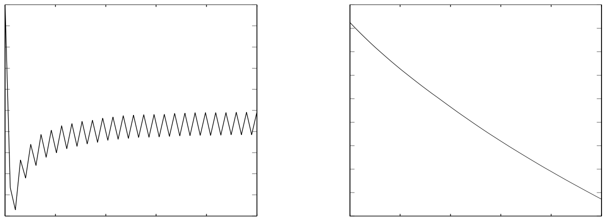


Observed image



Result image

We compared the efficiency of Newton algorithm to that of damped Newton algorithm.



Newton algorithm and Damped Newton algorithm

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