

## A NOTE ON THE ARTIN CONJECTURE

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ABSTRACT. In this paper, we survey some recent results on the Artin conjecture and discuss some aspects for the Artin conjecture.

### 1. INTRODUCTION

Let  $K/\mathbb{Q}$  be a Galois extension of  $\mathbb{Q}$  and  $\rho : \text{Gal}(K/\mathbb{Q}) \rightarrow GL(n, \mathbb{C})$  a non-trivial irreducible representation of its Galois group. E. Artin [1] associated to this data an  $L$ -function  $L(s, \rho)$ , defined for  $\text{Re } s > 1$ , which he conjectured to continue analytically to an entire function on the whole complex plane  $\mathbb{C}$  satisfying a functional equation. In 1947, R. Brauer [6] showed that the Artin  $L$ -function  $L(s, \rho)$  has a meromorphic continuation to a meromorphic function on  $\mathbb{C}$  and satisfies a functional equation.

Artin established his conjecture for the *monomial* representations, those induced from one-dimensional representation of a subgroup. His conjecture has not been solved yet in any dimension  $\geq 2$ . More evidence is provided in dimension 2 by R. Langlands, J. Tunnell, R. Taylor et al. In the case of two dimensional icosahedral representations, his conjecture still remains open. When  $\rho$  is an odd icosahedral representation, infinitely many examples of the Artin conjecture are known by the work of R. Taylor and others.

This article is organized as follows. In Section 2, we review the Galois representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  roughly. In Section 3, we describe the definition of the Artin  $L$ -function. In Section 4, we explain the connection between the Artin conjecture and the Langlands Functoriality Conjecture. In Section 5, we survey some known results on the Artin conjecture in the two dimensional case. In the final section we discuss some aspects for the Artin conjecture.

NOTATIONS: Throughout this paper,  $F$  denotes a number field,  $\overline{F}$  an algebraic closure of  $F$ , and  $\text{Gal}(\overline{F}/F)$  the absolute Galois group of  $F$ . We regard  $\text{Gal}(\overline{F}/F)$  as a topological group relative to the Krull topology. We write  $\mathbb{A}_F$  and  $I_F$  for the adèle ring and the idele group attached to  $F$  respectively. For each place  $v$  of  $F$ , we let  $F_v$  be the completion of  $F$  relative to  $v$ . We also fix an algebraic closure  $\overline{F}_v$  of  $F_v$  for each place  $v$ . For a square matrix  $A$ ,  $\text{tr}(A)$  denotes the trace of  $A$ .

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## 2. GALOIS REPRESENTATIONS

R. Taylor published a good survey paper [23] about Galois representations. The content of this section is a brief description of Section 1 in [23].

Let  $\mathbb{Q}$  be the field of rational numbers and  $\overline{\mathbb{Q}}$  denote the algebraic closure of  $\mathbb{Q}$ . We let  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  be the absolute Galois group of  $\mathbb{Q}$ . We see that  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  is a profinite topological group, a basis of open neighborhoods of the identity being given by the subgroups  $\text{Gal}(\overline{\mathbb{Q}}/K)$  as  $K$  runs over subextensions of  $\overline{\mathbb{Q}}/\mathbb{Q}$  which is finite over  $\mathbb{Q}$ . Let  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers, which is a totally disconnected locally compact topological field.  $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$  is an infinite extension of  $\mathbb{Q}_p$  and  $\overline{\mathbb{Q}}_p$  is not complete. We shall denote its completion by  $\mathbb{C}_p$ . Let  $\mathbb{Z}_p$  (resp.  $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ ) be the ring of integers in  $\mathbb{Q}_p$  (resp.  $\overline{\mathbb{Q}}_p$ ). These are local rings with maximal ideals  $p\mathbb{Z}_p$  and  $\mathfrak{m}_{\overline{\mathbb{Q}}_p}$  respectively. Then it is easy to see that the field  $\overline{\mathbb{F}}_p := \mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p}$  is an algebraic closure of the field  $\mathbb{F}_p := \mathbb{Z}_p/p\mathbb{Z}_p$ . Thus we obtain a continuous map

$$\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

which is surjective. Its kernel is called the *inertia subgroup* of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , and is denoted by  $I_{\mathbb{Q}_p}$ . The Galois group  $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$  is procyclic and has a canonical generator  $\text{Fr}_p$  called the Frobenius element defined by

$$\text{Fr}_p(x) = x^p, \quad x \in \overline{\mathbb{F}}_p.$$

I want to describe  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  via its representations. We have two natural representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , which are

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(n, \mathbb{C}), \quad \text{the Artin representations}$$

and

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(n, \overline{\mathbb{Q}}_l), \quad \text{the } l\text{-adic representations.}$$

Here  $GL(n, \overline{\mathbb{Q}}_l)$  is a group with  $l$ -adic topology. These representations are continuous.

The  $l$ -adic representations are closely related to an arithmetic geometry.

- A choice of embeddings  $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$  and  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$  establishes a bijection between isomorphism classes of Artin representations and isomorphism classes of  $l$ -adic representations with open kernel.
- There is a unique character

$$\chi_l : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}_l^\times \subset \overline{\mathbb{Q}}_l^\times$$

such that

$$\sigma\zeta = \zeta^{\chi_l(\sigma)}$$

for all  $l$ -power roots of unity  $\zeta$ . This is called the  *$l$ -adic cyclotomic character*.

- If  $X/\mathbb{Q}$  is a smooth projective variety, then the natural action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the cohomology

$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l) \cong H_{\text{et}}^i(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l)$$

is an  $l$ -adic representation.

We now discuss  $l$ -adic representations of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ . Let  $W_{\mathbb{Q}_p}$  be the subgroup of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  consisting of elements  $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  such that  $\sigma$  maps to  $\text{Fr}_p^{\mathbb{Z}} \subset \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ . We endow  $W_{\mathbb{Q}_p}$  with a topology by decreeing that  $I_{\mathbb{Q}_p}$  with its usual topology should be an open subgroup of  $W_{\mathbb{Q}_p}$ . We first consider the case  $l \neq p$ . We define a *WD-representation* of  $W_{\mathbb{Q}_p}$  over a field  $E$  to be a pair

$$r : W_{\mathbb{Q}_p} \longrightarrow GL(V), \quad \text{a continuous representation of } W_{\mathbb{Q}_p} \text{ with open kernel}$$

and

$$N \in \text{End}(V), \quad \text{a nilpotent endomorphism of } V$$

such that

$$r(\phi) N r(\phi^{-1}) = p^{-1} N$$

for every lift  $\phi \in W_{\mathbb{Q}_p}$  of  $\text{Fr}_p$ , where  $V$  is a finite dimensional  $E$ -vector space. A WD-representation  $(r, N)$  is said to be *unramified* if  $N = 0$  and  $r(I_{\mathbb{Q}_p}) = \{1\}$ . In the case  $E = \overline{\mathbb{Q}}_l$ , we call a WD-representation  $(r, N)$   *$l$ -integral* if all eigenvalues of  $r(\phi)$  has the absolute value 1. If  $l \neq p$ , then there is an equivalence of categories between  $l$ -integral WD-representations of  $W_{\mathbb{Q}_p}$  over  $\overline{\mathbb{Q}}_l$  and  $l$ -adic representations of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ . We will write  $WD_p(R)$  for the WD-representation associated to an  $l$ -adic representation  $R$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ . An  $l$ -adic representation  $R$  is said to be *unramified* if  $WD_p(R)$  is unramified. The case  $l = p$  is much more complicated because there are many more  $p$ -adic representations of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ . These have been extensively studied by J.-M. Fontaine et al. They single out certain special  $p$ -adic representations which are called *de Rham*. Indeed most  $p$ -adic representations are not de Rham. To any de Rham representation  $R$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  on a  $\overline{\mathbb{Q}}_p$ -vector space  $V$  they associate the following pair :

- A WD-representation  $WD_p(R)$  of  $W_{\mathbb{Q}_p}$  over  $\overline{\mathbb{Q}}_p$ .
- A multiset  $\text{HT}(R)$  of  $\dim V$  integers, called the Hodge-Tate numbers of  $R$ . The multiplicity of  $i$  in  $\text{HT}(R)$  is

$$\dim_{\overline{\mathbb{Q}}_p} (V \otimes_{\mathbb{C}_p} \mathbb{C}_p(i))^{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)},$$

where  $\mathbb{C}_p(i)$  denotes  $\mathbb{C}_p$  with  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -action and  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  acts on  $\mathbb{C}_p$  via  $\chi_p(\sigma)^i$  times its usual Galois action on  $\mathbb{C}_p$ .

We refer to [10, 11, 12] and [2] for more details on de Rham representations and their related materials.

We now discuss a so-called *geometric*  $l$ -adic representations. Fontaine and Mazur [13] proposed the following conjecture.

**Conjecture A** (Fontaine-Mazur) *Suppose that*

$$R : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(V)$$

*is an irreducible  $l$ -adic representation which is unramified at all but finitely many primes and with  $R|_{\text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}$  de Rham. Then there is a smooth projective variety  $X/\mathbb{Q}$  and integers  $i \geq 0$  and  $j$  such that  $V$  is a subquotient of  $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l(j))$ . In particular  $R$  is pure of some weight  $w \in \mathbb{Z}$ .*

Tate formulated the following conjecture.

**Conjecture B** (Tate). *Suppose that  $X/\mathbb{Q}$  is a smooth projective variety. Then there is a decomposition*

$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}) = \bigoplus_j M_j$$

with the following properties:

1. For each prime  $l$  and for each embeddings  $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ ,  $M_j \otimes_{\overline{\mathbb{Q}}, \iota} \overline{\mathbb{Q}}_l$  is an irreducible subrepresentation of  $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l)$ .
2. For all indices  $j$  and for all primes  $p$  there is a WD-representation  $WD_p(M_j)$  of  $W_{\mathbb{Q}_p}$  over  $\overline{\mathbb{Q}}_l$  such that

$$WD_p(M_j) \otimes_{\overline{\mathbb{Q}}, \iota} \overline{\mathbb{Q}}_l \cong WD_p(M_j \otimes_{\overline{\mathbb{Q}}, \iota} \overline{\mathbb{Q}}_l)$$

for all primes  $l$  and all embeddings  $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ .

3. There is a multiset of integers  $HT(M_j)$  such that
  - (a) for all primes  $l$  and all embeddings  $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$

$$HT(M_j \otimes_{\overline{\mathbb{Q}}, \iota} \overline{\mathbb{Q}}_l) = HT(M_j)$$

(b) and for all  $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$

$$\dim_{\mathbb{C}}(M_j \otimes_{\overline{\mathbb{Q}}, \iota} \mathbb{C}) \cap H^{a, i-a}(X(\mathbb{C}), \mathbb{C})$$

is the multiplicity of  $a$  in  $HT(M_j)$ .

If one believes conjecture A and B, then geometric  $l$ -adic representations should come in compatible families as  $l$  varies. There are many ways to make precise the notion of such a compatible family. See [23] for one of such family.

### 3. ARTIN $L$ -FUNCTIONS

Let  $F$  be a number field. We let

$$\sigma : \text{Gal}(\overline{F}/F) \longrightarrow GL(V)$$

be a finite dimensional Galois representation over  $F$ , where  $V$  is a finite dimensional complex vector space. The *Artin  $L$ -function*  $L(s, \sigma)$  attached to the Galois representation  $\sigma$  is defined to be an Euler product

$$L(s, \sigma) = \prod_v L(s, \sigma_v),$$

where  $v$  runs over all places of  $F$ . The local factor  $L(s, \sigma_v)$  is defined as follows. First we choose an embedding  $i_v : \overline{F} \longrightarrow \overline{F}_v$ , which gives rise to an embedding of Galois groups

$$j_v : \text{Gal}(\overline{F}_v/F_v) \longrightarrow \text{Gal}(\overline{F}/F)$$

via *restriction*. The composition  $\sigma_v = \sigma \circ j_v$  is a continuous representation of  $\text{Gal}(\overline{F}_v/F_v)$ . It depends on the choice of an embedding  $i_v$ , but different choices of  $i_v$  lead to conjugate embeddings  $j_v$ . So the equivalence class of  $\sigma_v$  is well defined and depends *only* on  $v$ .

In the nonarchimedean case, we let  $k_v$  and  $\overline{k_v}$  denote the residue fields of  $F_v$  and  $\overline{F_v}$  respectively.  $\text{Gal}(\overline{F_v}/F_v)$  acts on  $\overline{k_v}$  and we have an exact sequence

$$1 \longrightarrow I_v \longrightarrow \text{Gal}(\overline{F_v}/F_v) \longrightarrow \text{Gal}(\overline{k_v}/k_v) \longrightarrow 1,$$

where  $I_v$  is the inertia subgroup. We set  $q_v = |k_v|$ . A *Frobenius element*  $\text{Fr}_v$  is an element of  $\text{Gal}(\overline{F_v}/F_v)$  whose image in  $\text{Gal}(\overline{k_v}/k_v)$  is the automorphism

$$x \mapsto x^{q_v}, \quad x \in \overline{k_v}.$$

We note that the action of  $\sigma(\text{Fr}_v)$  on the subspace  $V^{I_v}$  of inertial invariants in  $V$  is independent of the choice of  $\text{Fr}_v$ . We define the local factor  $L(s, \sigma_v)$  at  $v$  by

$$L(s, \sigma_v) = \det \left( 1 - q_v^{-s} \sigma_v(\text{Fr}_v)|_{V^{I_v}} \right)^{-1}.$$

The Galois representation  $\sigma$  is said to be *unramified* at  $v$  if  $\sigma_v(I_v) = 1$ . In this case, the element  $\sigma_v(\text{Fr}_v)$  is independent of the choice of  $\text{Fr}_v$ . The *Frobenius class* attached to  $v$  is the conjugacy class  $\{\sigma_v(\text{Fr}_v)\}$  of  $\sigma_v(\text{Fr}_v)$  in  $GL(V)$ . The Frobenius class is independent of the choice of an embedding  $j_v$  and thus depends only on  $v$ . We note that it is a semisimple conjugacy class, that is, it consists of diagonalizable elements.

If  $v$  is archimedean, then  $F_v \cong \mathbb{R}$  or  $\mathbb{C}$ . In case  $F_v \cong \mathbb{R}$ , we get  $\text{Gal}(\overline{F_v}/F_v) \cong \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, c\}$ , where  $c$  denotes the complex conjugation. The eigenvalues of  $\sigma_v(c)$  are  $\pm 1$ . Let  $m_+$  (resp.  $m_-$ ) be the number of  $+1$  (resp.  $-1$ ) eigenvalues of  $\sigma_v(c)$ . In this case, we define the local factor  $L(s, \sigma_v)$  by

$$L(s, \sigma_v) = \left( \pi^{-s/2} \Gamma(s/2) \right)^{m_+} \left( \pi^{-(s+1)/2} \Gamma((s+1)/2) \right)^{m_-}.$$

If  $F_v \cong \mathbb{C}$ , then  $\text{Gal}(\overline{F_v}/F_v) \cong \{1\}$ . In this case, we define

$$L(s, \sigma_v) = \left( 2(2\pi)^{-s} \Gamma(s) \right)^n,$$

where  $n = \dim_{\mathbb{C}} V$ .

It is easy to see that

$$L(s, \sigma \oplus \tau) = L(s, \sigma) L(s, \tau)$$

for any two Galois representations  $\sigma$  and  $\tau$ . For any finite set  $S$  of places, we define the *partial L-function*  $L_S(s, \sigma)$  by

$$L_S(s, \sigma) = \prod_{v \notin S} L(s, \sigma_v).$$

We observe that if  $\sigma$  is the trivial representation and  $S$  is the archimedean places, then

$$L_S(s, \sigma) = \prod_{v < \infty} (1 - q_v^{-s})^{-1}$$

is nothing but the so-called Dedekind zeta function  $\zeta_F(s)$  of  $F$ .

## 4. THE ARTIN CONJECTURE AND FUNCTORIALITY

Since the eigenvalues of  $\sigma_v(\text{Fr}_v)$  at each place  $v$  are roots of unity, it is easy to see that the Euler product for  $L(s, \sigma)$  converges absolutely for  $\text{Re } s > 1$ . According to the works of E. Hecke [17], E. Artin [1] and R. Brauer [6], we obtain the following theorem.

**Theorem 4.1.** *Let  $F$  be a number field. Let  $\sigma : \text{Gal}(\overline{F}/F) \rightarrow GL(V)$  be a finite dimensional complex Galois representation over  $F$ . Then the Artin  $L$ -function  $L(s, \sigma)$  has a meromorphic continuation to a meromorphic function on  $\mathbb{C}$ . Moreover  $L(s, \sigma)$  satisfies a functional equation*

$$L(s, \sigma) = \epsilon(s, \sigma)L(1 - s, \sigma^*),$$

where  $\epsilon(s, \sigma)$  is the so-called epsilon factor (cf. [22]) and  $\sigma^*$  denotes the contragredient representation of  $\sigma$ .

**Artin Conjecture.** *If  $\sigma : \text{Gal}(\overline{F}/F) \rightarrow GL(V)$  is a nontrivial irreducible finite dimensional complex Galois representation over  $F$ , then the Artin  $L$ -function  $L(s, \sigma)$  can be analytically continued to an entire function on  $\mathbb{C}$ .*

Let  $F$  be a number field. For a cuspidal representation  $\pi$  of  $GL(n, \mathbb{A}_F)$ , we can define the automorphic  $L$ -function  $L(s, \pi)$  of  $\pi$  given by

$$L(s, \pi) = \prod_v L(s, \pi_v),$$

where  $v$  runs over all places of  $F$ . The precise definition of  $L(s, \pi_v)$  can be found in [5], [14], [15] and [19]. Jacquet and Langlands [18] proved that if  $n = 2$ ,  $L(s, \pi)$  can be analytically continued to an entire function on the whole complex plane  $\mathbb{C}$ . Godement and Jacquet [16] proved that for any positive integer  $n$ , the  $L$ -function  $L(s, \pi)$  can be analytically continued to an entire function on the whole complex plane  $\mathbb{C}$ .

R. Langlands proposed the following conjecture in order to attack the Artin conjecture.

**Langlands Functoriality Conjecture.** *Let  $F$  be a number field. Let*

$$\sigma : \text{Gal}(\overline{F}/F) \rightarrow GL(n, \mathbb{C})$$

*be a nontrivial irreducible finite dimensional complex Galois representation over  $F$ . Then there exists a cuspidal representation  $\pi(\sigma)$  of  $GL(n, \mathbb{A}_F)$  such that*

$$L(s, \sigma) = L(s, \pi(\sigma)).$$

We observe that if Langlands Functoriality Conjecture is true, by the work of Jacquet, Langlands and Godement, the Artin conjecture is true. The Langlands Functoriality Conjecture gave rise to some solutions of the Artin conjecture for an irreducible two-dimensional Galois representation.

I want to introduce the recent work of Andrew Booker. Let

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(2, \mathbb{C})$$

be an irreducible two-dimensional complex Galois representation over  $\mathbb{Q}$ . Booker [3] proved the following result.

**Theorem 4.2.** *If  $L(s, \rho)$  is not automorphic, then it has infinitely many poles. In particular, the Artin conjecture for  $\rho$  implies the Langlands Functoriality Conjecture for  $\rho$ .*

## 5. SPECIAL CASES OF THE ARTIN CONJECTURE

Let  $F$  be a number field. Let

$$\rho : \text{Gal}(\overline{F}/F) \longrightarrow GL(2, \mathbb{C})$$

be an irreducible two-dimensional complex Galois representation over  $F$ . The adjoint representation of the group  $GL(2, \mathbb{C})$  on the Lie algebra  $\mathfrak{gl}(2, \mathbb{C})$  induces the adjoint action of  $GL(2, \mathbb{C})$  on the three dimensional Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  of  $2 \times 2$  complex matrices of trace zero. We denote this representation by

$$\text{Ad} : GL(2, \mathbb{C}) \longrightarrow GL(3, \mathbb{C}).$$

The symmetric bilinear form  $\text{tr}(AB)$  is invariant under the adjoint action of  $GL(2, \mathbb{C})$ , and the image of  $\text{Ad}$  is isomorphic to the complex orthogonal group  $SO(3, \mathbb{C})$  defined by this bilinear form  $\text{tr}(AB)$ . Irreducible two-dimensional representations are classified according to the image of  $\text{Ad} \circ \rho$  in  $SO(3, \mathbb{C})$ . It is known that a finite subgroup of  $SO(3, \mathbb{C})$  is either *cyclic*, *dihedral* or isomorphic to one of the symmetry groups of the Platonic solids :

- (1) *tetrahedral group*  $\cong A_4$  ;
- (2) *octahedral group*  $\cong S_4$  ;
- (3) *icosahedral group*  $\cong A_5$ .

We shall say that  $\rho$  is of *cyclic*, *dihedral*, *tetrahedral*, *octahedral*, *icosahedra type* if the image of  $\text{Ad} \circ \rho$  in  $SO(3, \mathbb{C})$  is of the corresponding type. The Artin conjecture was solved by E. Artin for the cyclic and dihedral type, by R. Langlands [20] for the tetrahedral type, and was solved completely by J. Tunnell [25, 26] for the octahedral type. Indeed we can show that  $\pi(\rho)$  exists if  $\rho$  is of cyclic, dihedral, tetrahedral or octahedral type. We refer to [21] for sketchy proofs in such these types. The Artin conjecture for the *icosahedral type* has not been solved yet, although it has been verified in few very special cases [7, 8, 9, 24].

Recently Taylor et al gave some evidences to the Artin conjecture for the icosahedral type. K. Buzzard, M. Dickinson, N. Sheherd-Baron and R. Taylor [8] proved that the Artin conjecture is true for certain special odd icosahedral representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  by showing that they are modular. I describe this content explicitly.

**Theorem 5.1.** *Suppose that  $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(2, \mathbb{C})$  is a continuous representation and that  $\rho$  is odd, i.e., the determinant of  $\rho(c)$  is  $-1$ , where  $c$  is the complex conjugation. If  $\rho$  is of icosahedral type, we assume that*

- *the projectivised representation  $\text{proj}(\rho) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}(2, \mathbb{C})$  is unramified at 2*  
*and that image of a Frobenius element at 2 under  $\text{proj}(\rho)$  has order 3,*
- *and  $\text{proj}(\rho)$  is unramified at 5.*

*Then there is a new form of weight one such that for all primes  $p$  the  $p$ -th Fourier coefficient of  $f$  equals the trace of Frobenius at  $p$  on the inertia at  $p$  covariants of  $\rho$ . In particular the Artin  $L$ -function for  $\rho$  is the Mellin transform of a newform of weight one and is an entire function.*

Moreover R. Taylor [24] proved the following.

**Theorem 5.2.** *Let  $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}(2, \mathbb{C})$  is a continuous representation and that  $\rho$  is odd, i.e., the determinant of  $\rho(c)$  is  $-1$ , where  $c$  is the complex conjugation. If  $\rho$  is of icosahedral type, we assume that the projective image of the inertia group at 3 has odd order and the projective image of the decomposition group at 5 is unramified at 2. Then  $\rho$  is modular and its Artin  $L$ -function  $L(s, \rho)$  is entire.*

## 6. FINAL REMARKS

As mentioned before, we still have no idea of verifying the Artin conjecture for  $n$ -dimensional Galois representations with  $n \geq 3$ . In the case of two dimensional icosahedral representations, the Artin conjecture still remains open. If the Artin conjecture is true for a certain Galois representation  $\rho$ , it might be interesting to find methods for locating zeros of the Artin  $L$ -function  $L(s, \rho)$ . In [4], A. Booker discusses two methods for locating zeros of  $L(s, \rho)$ . He also presents a group-theoretic criterion under which one may verify the Artin conjecture for some non-monomial Galois representations, up to finite height in the complex plane.

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