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NOTES ON CERTAIN REAL ABELIAN 2-EXTENSION FIELDS WITH $\lambda_2 = \mu_2 = \nu_2 = 0$

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Abstract. In this note, we will report our recent results on certain real abelian 2-extension fields with Iwasawa invariants $\lambda_2 = \mu_2 = \nu_2 = 0$. We also recall some of previous results on certain real abelian fields with Iwasawa invariants $\lambda_p = \mu_p = \nu_p = 0$ in the case of odd prime numbers p for convenience of the readers.

1. Introduction

For a number field k and a prime number p, we denote by $A(k)$ the p-Sylow subgroup of the ideal class group of k, by k_∞ the cyclotomic $\mathbb{Z}_p$-extension of k, and by k_n the n-th layer in k_∞/k. Then Iwasawa [Iw59] proved that there exist three integers $\lambda = \lambda_p(k) \geq 0$, $\mu = \mu_p(k) \geq 0$ and $\nu = \nu_p(k)$, depending only on k and p, such that

$$\#A(k_n) = p^{\lambda n + \mu p^n + \nu}$$

for every sufficiently large n, where # means the cardinality of a subsequent object. These integers $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ are called the (cyclotomic) Iwasawa λ-, μ- and ν-invariants, respectively, of k for p. We denote these invariants simply by λ_p, μ_p and ν_p when the base field we consider is clear.

Concerning these invariants, Iwasawa mentions that it would be an important problem to find out if $\lambda_p(\mathbb{Q}(\zeta_p)^+) = \mu_p(\mathbb{Q}(\zeta_p)^+) = 0$ for any prime number p in [Iw70, page 392], or to find out when the "plus-part" of $\lambda_p(k)$ is positive for CM-fields k in [Iw73, page 316], where $\mathbb{Q}(\zeta_p)^+$ is the maximal real subfield of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$, and the "plus-part" is the part of $\lambda_p(k)$ corresponding to the maximal real subfield. Following this, Greenberg proposed in [Gr76] that it would be $\lambda_p(k) = \mu_p(k) = 0$ for any totally real number field k and any prime number p. This is now known as Greenberg's conjecture.

As for the μ-invariants, Ferrero and Washington [FW79] proved that $\mu_p(k) = 0$ for any abelian (not necessarily totally real) number field k and any prime number p. As for Greenberg's conjecture, we have infinitely many examples of real abelian fields for which the conjecture is true (for example, see Byeon [By01], Nakagawa-Horie [NH88], Ozaki-Taya [OT97], Taya [Ta00] and Yamamoto [Ya00] and so on), and also have some nice criteria for $\lambda_p(k)$ to be zero when k is a real abelian number field and p is an odd prime number (for example, see Fukuda-Komatsu

[FK86], Fukuda-Taya [FT95], Ichimura-Sumida [IS96-7], Kraft-Schoof [KS95], Kurihara [Ku99], Tsuji [Ts03] and the papers referred there). However, we do not still know so much about the λ-invariants for totally real number fields k except for $k = \mathbb{Q}$, even in the case of real quadratic fields. Actually, the field $\mathbb{Q}$ of rational numbers is the only example for which we know that $\lambda_p = \mu_p = 0$ for any prime number p at present.

In the case of $\mathbb{Q}$, we further know that $\lambda_p(\mathbb{Q}) = \mu_p(\mathbb{Q}) = \nu_p(\mathbb{Q}) = 0$ for any prime number p. So, it would be an interesting research to study abelian fields with $\lambda_p = \mu_p = \nu_p = 0$. In the previous work [Ya00], the second author determined all real abelian p-extensions k over $\mathbb{Q}$ with $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ for any odd prime number p. On the other hand, in the papers [NH88] and [Ta00], Nakagawa-Horie and the first author gave some non-zero lower bound for the density of real quadratic fields k with $\lambda_3(k) = \mu_3(k) = \nu_3(k) = 0$ in all real quadratic fields. In this note, we report our recent results in the case where $p = 2$ together with some of known related results. We first deal with a determination problem for $p = 2$ in § 2, and next with a density problem for $p = 2$ in § 3, as compared with the case of odd prime numbers.

2. Determination of real abelian 2-extension fields with $\lambda_2 = \mu_2 = \nu_2 = 0$

In this part, we determine all real abelian 2-extensions k over $\mathbb{Q}$ with $\lambda_2 = \mu_2 = \nu_2 = 0$. For any odd prime number p, all real abelian p-extensions k over $\mathbb{Q}$ with $\lambda_p = \mu_p = \nu_p = 0$ are determined by the second author [Ya00].

Firstly, we recall the result on the case of odd primes. Let p be an odd prime number. We denote by $\left(\frac{\cdot}{\cdot}\right)_p$ the p-th power residue symbol. Namely, $\left(\frac{x}{q}\right)_p = 1$ means that x is the p-th power of some integer modulo q, where q is a prime number with $q \equiv 1 \pmod p$ and x is an integer prime to q. Let k be an abelian p-extension of $\mathbb{Q}$. Then it is easily seen that the conductor f_k of k can be written as $f_k = p^a p_1 \cdots p_t$, where a is a non-negative integer and $p_1, \cdots, p_t$ are distinct primes with $p_i \equiv 1 \pmod p$. Let k_G be the genus p-class field of $k/\mathbb{Q}$. Then we have the following result:

Theorem 2.1 ([Ya00]). *Let p be an odd prime number and k an abelian p-extension of $\mathbb{Q}$ with conductor $f_k = p^a p_1 \cdots p_t$, where a is a non-negative integer and $p_1, \cdots, p_t$ are distinct primes. If*

$$(*) \qquad \lambda_p(k) = \mu_p(k) = \nu_p(k) = 0,$$

then $t \leq 2$. Conversely, we assume that $t \leq 2$. Then we have the following:

(i) If $t = 0$, then $(*)$ holds.

(ii) In the case of $t = 1$: The condition $(*)$ holds if and only if $k_G \subseteq k_\infty$, and

$$\left(\frac{p}{p_1}\right)_p \neq 1 \text{ or } p_1 \not\equiv 1 \pmod{p^2}.$$

(iii) In the case of $t = 2$: The condition $(*)$ holds if and only if $k_G \subseteq k_\infty$, and for $(i, j) = (1, 2)$ or $(2, 1)$,

$$\left(\frac{p}{p_i}\right)_p \neq 1, \ \left(\frac{p_i}{p_j}\right)_p \neq 1, \ p_j \not\equiv 1 \pmod{p^2},$$

and, there exist $x, y, z \in \mathbb{Z}$ such that

$$\left(\frac{p_j p^x}{p_i}\right)_p = 1, \ \left(\frac{p p_i^y}{p_j}\right)_p = 1, \ p_i p_j^z \equiv 1 \pmod{p^2}, \ \text{and } xyz \not\equiv -1 \pmod{p}.$$

Remark 2.2. *One can show that there exist infinitely many fields satisfying each of the conditions (i)∼(iii) of Theorem 2.1 by Chebotarev density theorem (cf. Lemma 4 in* [Iw89]*).*

Example 2.3. (In the case of $p = 3$) *There exist 611 cubic cyclic fields whose conductors are prime (i.e., $a = 0$ and $t = 1$ in Theorem 2.1) less than 10^4. Then 547 fields of them satisfy the condition (ii) in Theorem 2.1, so $\lambda_3 = \mu_3 = \nu_3 = 0$. On the other hand, as for Greenberg's conjecture, it is known that $\lambda_3 = \mu_3 = 0$ for all 611 cubic cyclic fields whose conductors are prime numbers less than 10^4 (cf.* [FK01]*).*

In the case of $t = 2$ in Theorem 2.1, there exist 305 fields k which satisfy $k = k_G$ and $f_k = p_1 p_2 < 10^4$, where $p_1 \equiv p_2 \equiv 1 \pmod 3$ are distinct prime numbers. Then 179 fields of them satisfy the condition (iii) in Theorem 2.1, so $\lambda_3 = \mu_3 = \nu_3 = 0$.

Next, we consider the case of $p = 2$. In this case, the similar idea in [Ya00] is applicable, but more complicated than the case where p is odd. In fact, the infinite place ∞ of $\mathbb{Q}$ is unramified in an abelian p-extension field, where p is an odd prime number, but may be ramified in an abelian 2-extension field. This is a reason why the case of $p = 2$ (i.e., the description of Theorem 2.4) is more complicated than the case of odd primes (i.e., Theorem 2.1).

Let k be a real abelian 2-extension field over $\mathbb{Q}$ and k_∞ the cyclotomic $\mathbb{Z}_2$-extension of k with n-th layer k_n. We denote by f_k the conductor of k as before. For the simplicity, we assume that f_k is not divisible by 8. This assumption is not essential: Let $k/\mathbb{Q}$ be an abelian 2-extension with $8 \mid f_k$. Then, we see easily that there exists an abelian 2-extension field F such that $8 \nmid f_F$ and $k_\infty = F_\infty$. Namely, any real abelian 2-extension field k is always contained the cyclotomic $\mathbb{Z}_2$-extensions of such an abelian 2-extension field F whose conductor not dividing 8. Of course, both Iwasawa invariants of k_∞ and F_∞ are coincide.

Since f_k is not divisible by 8, all primes ramified in k_∞/k are totally ramified. Then, we get easily that $\#A(k_n) \leq \#A(k_{n+1})$ for all integers $n \geq 0$. Hence $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$ if and only if $A(k_n) = 0$ for all integers $n \geq 0$. By a theorem of Fukuda [Fu94], the latter assertion is equivalent to $A(k_1) = 0$. We denote by k_G (*resp.* k_C) the genus 2-class field (*resp.* the central 2-class field) of $k/\mathbb{Q}$. Then it follows easily that $A(k_1) = 0$ is equivalent to $k_{1,C} = k_{1,G} = k_1$. Therefore, in order to determine all real abelian 2-extensions k over $\mathbb{Q}$ with $\lambda_2 = \mu_2 = \nu_2 = 0$, we should investigate the structure of the Galois extension of $k_{1,C}/k_{1,G}$ and $k_{1,G}/k_1$.

Now we prepare some notation. For an odd prime number p and an integer e which divides $p-1$, we denote by $\mathbb{Q}^{(e)}(p)$ the unique subfield of $\mathbb{Q}(\zeta_p)$ of degree e over $\mathbb{Q}$. For example, $\mathbb{Q}^{(2)}(p) = \mathbb{Q}(\sqrt{p^*})$, where $p^* = (-1)^{\frac{p-1}{2}}p$. Also, let v_2 be the (additive) 2-adic valuation such that $v_2(2) = 1$ and put $n_p = v_2(p-1)$.

The results of [Fr83] tell us that $\mathrm{Gal}(k_{1,C}/k_{1,G})$ can be computed by some information of primes ramified in $k/\mathbb{Q}$. Also, for a given abelian 2-extension field k, $k_{1,G}$ can be computed easily. Hence, we have the following result. The detail of the proof and more precise information of it will published somewhere.

Theorem 2.4. *Let $k/\mathbb{Q}$ be a real abelian 2-extension with $8 \nmid f_k$. Then*

$$\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$$

if and only if k is one of the following fields:

(i) *Let p be a prime number.*
  (i-1) $p \equiv 3 \pmod 4$; $k = \mathbb{Q}(\sqrt{p})$,
  (i-2) $p \equiv 5 \pmod 8$; $k = \mathbb{Q}(\sqrt{p})$ *or* $(\mathbb{Q}(\sqrt{-1})\mathbb{Q}^{(4)}(p))^+$,
  (i-3) $p \equiv 1 \pmod 8$, $\left(\frac{2}{p}\right)_4\left(\frac{p}{2}\right)_4 = -1$; $k = \mathbb{Q}^{(e)}(p)$, $e = 2^1, 2^2, \cdots, 2^{n_p-1}$.

(ii) *Let p and q be distinct prime numbers.*
  (ii-1) $p \equiv 3 \pmod 4, q \equiv 3 \pmod 8$; $k = \mathbb{Q}(\sqrt{pq})$ *or* $\mathbb{Q}(\sqrt{p}, \sqrt{q})$,
  (ii-2) $p \equiv 3 \pmod 4, q \equiv 5 \pmod 8$;
  $k = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ *or* $(\mathbb{Q}(\sqrt{-1}, \sqrt{p})\mathbb{Q}^{(4)}(q))^+$ *or* $(\mathbb{Q}(\sqrt{-p})\mathbb{Q}^{(4)}(q))^+$,
  (ii-3) $p \equiv q \equiv 5 \pmod 8$, $\left(\frac{q}{p}\right) = 1$, $\left(\frac{q}{p}\right)_4\left(\frac{p}{q}\right)_4 = -1$; $k = \mathbb{Q}(\sqrt{p}, \sqrt{q})$,
  (ii-4) $p \equiv q \equiv 5 \pmod 8$, $\left(\frac{q}{p}\right) = -1$, $\left(\frac{2q}{p}\right)_4\left(\frac{2p}{q}\right)_4\left(\frac{pq}{2}\right)_4 = -1$; $k = \mathbb{Q}(\sqrt{p}, \sqrt{q})$,
  (ii-5) $p \equiv 5 \pmod 8, q \equiv 1 \pmod 8$, $\left(\frac{q}{p}\right) = -1$, $\left(\frac{2}{q}\right)_4\left(\frac{q}{2}\right)_4 = -1$;
  $k = \mathbb{Q}(\sqrt{p})\mathbb{Q}^{(e)}(q)$, $e = 2^1, 2^2, \cdots, 2^{n_q-1}$.

(iii) *Let p, q and r be distinct prime numbers.*
  (iii-1) $p \equiv q \equiv r \equiv 3 \pmod 8$; $k = \mathbb{Q}(\sqrt{pq}, \sqrt{pr})$,
  (iii-2) $p \equiv q \equiv 3, r \equiv 5 \pmod 8$, $\left(\frac{pq}{r}\right) = -1$;
  $k = \mathbb{Q}(\sqrt{pq}, \sqrt{r})$ *or* $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$ *or* $(\mathbb{Q}(\sqrt{-p}, \sqrt{-q})\mathbb{Q}^{(4)}(r))^+$ *or* $(\mathbb{Q}(\sqrt{-1}, \sqrt{p}, \sqrt{q})\mathbb{Q}^{(4)}(r))^+$,
  (iii-3) $p \equiv q \equiv 3, r \equiv 7 \pmod 8$; $k = \mathbb{Q}(\sqrt{pq}, \sqrt{pr})$,
  (iii-4) $p \equiv q \equiv 3, r \equiv 7 \pmod 8$, $\left(\frac{pq}{r}\right) = -1$; $k = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$,
  (iii-5) $p \equiv 3$, $q \equiv 7$, $r \equiv 5 \pmod 8$, $\left(\frac{q}{r}\right) = -1$;
  $k = \mathbb{Q}(\sqrt{pq}, \sqrt{r})$ *or* $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$ *or* $(\mathbb{Q}(\sqrt{-p}, \sqrt{-q})\mathbb{Q}^{(4)}(r))^+$ *or* $(\mathbb{Q}(\sqrt{-1}, \sqrt{p}, \sqrt{q})\mathbb{Q}^{(4)}(r))^+$.

Here, $\left(\frac{\cdot}{\cdot}\right)_4$ is the biquadratic residue character (cf [Fr83, p.76]*), and for a imaginary abelian number field K, K^+ means the maximal real subfield of K.*

Now we make a few remarks on the biquadratic residue character. In the case of $a \equiv 1 \pmod 8$, only where we need in the above theorem, $\left(\frac{a}{2}\right)_4$ is easily calculated as follows:

$$\left(\frac{a}{2}\right)_4 = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{16}, \\ -1 & \text{if } a \equiv 9 \pmod{16}. \end{cases}$$

Also, in the case of $p \equiv 1 \pmod 4$, which is prime, and of $a \equiv 1 \pmod 4$, $\left(\frac{a}{p}\right)_4$ can be calculated as follows:

$$\left(\frac{a}{p}\right)_4 = \begin{cases} 1 & \text{if } a^{\frac{p-1}{4}} \equiv 1 \pmod p, \\ -1 & \text{if } a^{\frac{p-1}{4}} \not\equiv 1 \pmod p. \end{cases}$$

Remark 2.5. *As in the case where p is odd prime number, one can show that there exist infinitely many fields satisfying each conditions of Theorem 2.4, (i-1) to (iii-5), by Chebotarev density theorem.*

3. The density of real quadratic fields with $\lambda_2 = \mu_2 = \nu_2 = 0$

In this section, we deal with results on the density of real quadratic fields k with $\lambda = \mu = \nu = 0$ in all real quadratic fields. For a real number x, we denote by $\mathcal{K}^+(x)$ the set of real quadratic fields k with discriminant d_k less than x.

First, we recall some results in the case of $p = 3$. In this case, we do not determine all real quadratic field with $\lambda_3(k) = \mu_3(k) = \nu_3(k) = 0$. However, we have a nice result to estimate the density of real quadratic fields whose class number is not divisible by 3. Actually, Nakagawa and Horie refined a result of Davenport and Heilbronn [DH71], concerning the theory of binary cubic form, by attaching congruence conditions to discriminants, and gave the following result:

Theorem 3.1 (Nakagawa-Horie [NH88])**.** *We have*

$$\liminf_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x)\ \mid\ 3 \nmid h(k),\ d_k \equiv 2 \pmod 3\ \}}{\#\mathcal{K}^+(x)} \geq \frac{5}{16},$$

and

$$\liminf_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x)\ \mid\ 3 \nmid h(k),\ d_k \equiv 0 \pmod 3\ \}}{\#\mathcal{K}^+(x)} \geq \frac{5}{24}.$$

Here, $h(k)$ is the class number of k and d_k is the discriminant of k.

In the case where $p = 3$ does not split in k, a theorem of Iwasawa [Iw56] on indivisibility of class numbers of relative p-extensions says that if the class number of k is not divisible by 3, then $\lambda_3(k) = \mu_3(k) = \nu_3(k) = 0$. Hence, the following corollary is an immediate consequence of Theorem 3.1

Corollary 3.2. *We have*

$$\liminf_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x)\ \mid\ \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0,\ d_k \equiv 2 \pmod 3\ \}}{\#\mathcal{K}^+(x)} \geq \frac{5}{16},$$

and

$$\liminf_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x)\ \mid\ \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0,\ d_k \equiv 0 \pmod 3\ \}}{\#\mathcal{K}^+(x)} \geq \frac{5}{24}.$$

Remark 3.3. *We can obtain a similar type of corollary as above in the case of imaginary quadratic fields (for the statement, see* [NH88]*).*

In the case where $p = 3$ splits, we can not apply the above theorem of Iwasawa on indivisibility of class numbers to show $\lambda_3 = \mu_3 = \nu_3 = 0$. However, we can get the following fortunately:

Proposition 3.4 ([Ta00])**.** *Let $d > 0$ be a square-free integer with $d \equiv 1 \pmod 3$. Put $k = \mathbb{Q}(\sqrt{d})$ and $k^* = \mathbb{Q}(\sqrt{-3d})$. Then the following are equivalent:*

(i) $\lambda_3(k) = \mu_3(k) = \nu_3(k) = 0$,

(ii) *the class number of k^* is not divisible by* 3.

From this proposition, we can reduce finding real quadratic fields k with $\lambda_3(k) = \mu_3(k) = \nu_3(k) = 0$ and with the prime 3 splitting to finding corresponding imaginary quadratic fields k^* whose class number is not divisible by 3. Therefore, we obtain the following as a corollary to Proposition 3.4, by applying a refinement of a result of Davenport and Heilbronn [DH71] due to Nakagawa and Horie [NH88] to imaginary quadratic fields k^*, even in the case where $p = 3$ splits in k.

Corollary 3.5. *We have*

$$\liminf_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x)\ \mid\ \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0,\ d_k \equiv 1 \pmod 3\ \}}{\#\mathcal{K}^+(x)} \geq \frac{3}{16}.$$

Now, we have the following theorem by combining two corollaries above.

Theorem 3.6 (Nakagawa-Horie [NH88] and Taya [Ta00])**.** *We have*

$$\liminf_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x)\ \mid\ \lambda_3(k) = \mu_3(k) = \nu_3(k) = 0\ \}}{\#\mathcal{K}^+(x)} \geq \frac{17}{24}.$$

Theorem 3.6 tells us that we have real quadratic fields with $\lambda_3 = \mu_3 = \nu_3 = 0$ in the proportion of at least seven to ten. In particular, we see that there exist infinitely many real quadratic fields k with $\lambda_3 = \mu_3 = \nu_3 = 0$, even if we further set a prime decomposition condition of the prime 3 in k to k by Corollaries 3.2 and 3.5.

Well, how about the other primes $p \neq 3$? In any case, is such a proportion positive?

Next, we consider the case where $p = 2$. In this case, we can obtain a result on the density of quadratic fields with $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$, by using a result for $p = 2$ in the previous section. First, Theorem 2.4 implies the following proposition, which contains real quadratic fields with conductor divisible by 8:

Proposition 3.7. *Let k be a real quadratic field. Then $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$ if and only if k is one of the following:*

(i) $k = \mathbb{Q}(\sqrt{p})$*, where p is a prime number such that*

  (i-1) $p = 2$, *or*

  (i-2) $p \equiv 3 \pmod 4$, *or*

  (i-3) $p \equiv 5 \pmod 8$, *or*

  (i-4) $p \equiv 1 \pmod 8$ *and* $\left(\frac{2}{p}\right)_4 \left(\frac{p}{2}\right)_4 = -1$,

(ii) $k = \mathbb{Q}(\sqrt{2p})$*, where p is a prime number satisfying (i-2), (i-3) or (i-4),*

(iii) $k = \mathbb{Q}(\sqrt{pq})$, *where p and q are distinct prime numbers such that*
  (iii-1) $p \equiv q \equiv 3 \pmod 8$, *or*
  (iii-2) $p \equiv 7,\ q \equiv 3 \pmod 8$,

(iv) $k = \mathbb{Q}(\sqrt{2pq})$, *where p and q are distinct prime numbers satisfying (iii-1) or (iii-2).*

The proportion of real quadratic fields listed in the proposition above can be estimated by analytic number theory. Hence, we can get the following theorem. Though the proof is omitted here, we will publish it somewhere.

Theorem 3.8. *We have*

$$\lim_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^+(x) \mid \lambda_2(k) = \mu_2(k) = \nu_2(k) = 0\ \}}{\#\mathcal{K}^+(x)} = 0.$$

Theorem 3.8 says that the density in the case of $p = 2$ is zero. So, the behavior in the case where $p = 2$ is completely different from the one in the case where $p = 3$.

Here we also make a few remarks about the case where $p = 2$. First, we see by Proposition 3.7 that there exist infinitely many real quadratic fields k with $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$ (see also [OT97]), though the density of such real quadratic fields is equal to zero. Next, we can prove Proposition 3.7 directly in an elementary way without Theorem 2.4. Finally, in the case of imaginary quadratic fields, we easily see the following by so-called Kida's formula (cf. [Ki79]) and analytic number theory.

Remark 3.9. *Let k be an imaginary quadratic field. Then $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$ if and only if k is one of the following:*

(i) $k = \mathbb{Q}(\sqrt{-1})$,

(ii) $k = \mathbb{Q}(\sqrt{-2})$,

(iii) $k = \mathbb{Q}(\sqrt{-p})$, *where p is a prime number satisfying $p \equiv 3 \pmod 8$,*

(iv) $k = \mathbb{Q}(\sqrt{-2p})$, *where p is a prime number satisfying the condition in (iii).*

And further, $\lambda_2(k) = \mu_2(k) = 0$ and $\nu_2(k) > 0$ if and only if k is one of the following:

(v) $k = \mathbb{Q}(\sqrt{-p})$, *where p is a prime number satisfying $p \equiv 5 \pmod 8$,*

(vi) $k = \mathbb{Q}(\sqrt{-2p})$, *where p is a prime number satisfying the condition in (v).*

Therefore, it follows that

$$\lim_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^-(x) \mid \lambda_2(k) = \mu_2(k) = \nu_2(k) = 0\ \}}{\#\mathcal{K}^-(x)} = 0,$$

and

$$\lim_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^-(x) \mid \lambda_2(k) = \mu_2(k) = 0,\ \nu_2(k) > 0\ \}}{\#\mathcal{K}^-(x)} = 0,$$

where, for a real number x, $\mathcal{K}^-(x)$ denotes the set of imaginary quadratic fields k with discriminant d_k more than x. In particular, we have

$$\lim_{x\to\infty} \frac{\#\{\ k \in \mathcal{K}^-(x) \mid \lambda_2(k) = \mu_2(k) = 0\ \}}{\#\mathcal{K}^-(x)} = 0.$$

From this, as for Greenberg's conjecture, the assumption that k is totally real is essential.

On the other hand, in the case where $p \geq 5$, several researchers investigate the indivisibility of class numbers of quadratic fields, by using class numbers relation formula (cf. Hartung [Ha74] in the case of imaginary), by using Eichler's trace formula and Eichler-Shimura relation (cf. Horie [Ho87], Horie-Ônishi [HO88] and so on in the case of imaginary), and by using modular forms of half-integral weight (cf. Kohnen-Ono [KO99], Byeon [By99] and so on in the case of imaginary, and Ono [On99], Byeon [By01] and so on in the case of real). And, some of them can be applied to a problem on the vanishing of Iwasawa invariants (see Horie [Ho87] and Byeon [By99], [By01]).

In particular, the method by using modular forms of half-integral weight enables us to estimate the number of quadratic fields whose class number is not divisible by p, or satisfying $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$, at least for a given prime number $p \geq 5$. However, the estimate of lower bound of the number of such quadratic fields obtained by this method is relatively small in comparison with the number of all quadratic fields. So, it is not enough to get information on the density of such quadratic fields for prime numbers $p \geq 5$, as of now.

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