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p-ADIC L-FUNCTIONS FOR GALOIS DEFORMATIONS AND IWASAWA MAIN CONJECTURE

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Contents

In my talk on 6th January at Japan-Korea Number theory seminar, I explained about the analytic p-adic L-function $L_p(\mathcal{T})$ associated to a family $\mathcal{T}$ of p-adic Galois representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a p-adic rigid analytic space $\mathcal{B}$. The existence and the characterization of $L_p(\mathcal{T})$ satisfying an appropriate interpolation property seems not completely known even conjecturally. Unless $\mathcal{T}$ has a good local property as a representation of the decomposition group at p, the existence of $L_p(\mathcal{T})$ seems hopeless. Even when it seems that $L_p(\mathcal{T})$ should exist, the characterization of $L_p(\mathcal{T})$ is not so clear. As far as I understand, difficulties on (conjectural) characterization of $L_p(\mathcal{T})$ are concerned with problems on complex periods and p-adic periods. in another proceeding article [O6], we already summarized some of our perspective on a generalization of the Iwasawa theory extending the idea first proposed by Greenberg [Gr3] (see also the introduction of [O5]). In this paper, we also discuss our program, but from a slightly different point of view from those in the above mentioned references. In the latter half of this article, we also treat the first non-trivial realization of our general program in the case of nearly ordinary two-variable Hida deformations. In this case, most of expected results for positive evidence of the generalize Iwasawa theory are now proved in the references [O3], [O4], [O5]. We would like to refer to these references for the detail which can not be covered in this short article.

1. A quick tour on several classical analytic p-adic L-functions

Let p be an odd prime number, which will be fixed throughout the paper. p-power congruence and p-adic continuity is an useful idea to study special values of L-functions or other important objects in arithmetic geometry. We denote by

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Γ the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$, which is the unique Galois extension of the rational number field $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_{p^\infty})$ with χ_{cyc} : $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \overset{\sim}{\longrightarrow} 1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ via the p-adic cyclotomic character χ_{cyc}. We will fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, where $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}}_p$) is a fixed algebraic closure of $\mathbb{Q}$ (resp. the p-adic field $\mathbb{Q}_p$) and $\mathbb{C}$ is the field of complex numbers.

The first basic result in p-adic study of global L-functions is the following analytic p-adic L-functions associated to Dirichlet characters:

Theorem 1.1 (Kubota-Leopoldt, Iwasawa, Coleman)**.** *Let D be a natural number prime to p and let ψ be a non-trivial Dirichlet character with conductor D and satisfying $\psi(-1) = 1$. Then, there exists an element $L_p(\psi) \in \mathbb{Z}_p[\psi][[\Gamma]]$ with the following interpolation property:*

$$\chi_{\mathrm{cyc}}^r(L_p(\psi)) = (1 - p^{r-1}\psi(p))L(\psi, 1-r),$$

where r runs non negative integers divisible by $p-1$.

Remark 1.2.
(1) Note that any continuous character $\eta : \Gamma \longrightarrow \overline{\mathbb{Q}}_p^\times$ induces naturally an algebra homomorphism $\mathcal{O}[[\Gamma]] \longrightarrow \overline{\mathbb{Q}}_p$ for the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$.
(2) The spacial value $L(\psi, 1-r) \in \mathbb{C}$ is known to be a non-zero algebraic number for every non-negative r divided by $p-1$. Thus, the equation in the interpolation property makes sense in $\overline{\mathbb{Q}}_p$.
(3) The condition on D and ψ in the above theorem is not really necessary for the existence of "p-adic L-function". It is only for avoiding the complicated statement and for being a quick introduction that we did not state the theorem under the most general assumption. We also have the natural interpolation at every non-negative integer r without the above congruence, but we avoided a general statement from the same reason as above.

If we do not hesitate the use of the terminology of "motive", the above analytic p-adic L-function corresponds to a rank-one motive $M(\psi)$ (or the p-adic Galois representation for $M(\psi)$) defined over $\mathbb{Q}$. We have also the algebraic p-adic L-function for $M(\psi)$. Though we do not discuss the algebraic p-adic L-function in this article, the relation between the algebraic p-adic L-function and the analytic p-adic L-function which is later called as "the Iwasawa Main conjecture" strongly motivated the active research of the Iwasawa theory for $M(\psi)$.

By successful motivation of the Iwasawa theory for $M(\psi)$, it was natural that people tried to study the Iwasawa theory for more general motives. The second basic motive following $M(\psi)$ is rank-two motives associated to elliptic curves. Mazur and several others studied this case seriously and they realized that the generalization of the Iwasawa theory to elliptic curves E is hopeful if E has ordinary reduction at p. The analytic p-adic L-function for this case is as follows:

Theorem 1.3 (Mazur and Swinnerton-Dyer)**.** *Let E be an elliptic curve which has good ordinary reduction at p. Then there exists an element $L_p(E) \in \mathbb{Z}_p[[\Gamma]]$ with the following interpolation properties:*

$$\phi(L_p(E)) = \left(1 - \frac{\phi(p)}{\alpha}\right)^2 \alpha^{-s(\phi)} G(\phi^{-1}) \frac{L(E, \phi, 1)}{\Omega_E^+},$$

where $\Omega_E^+ \in \mathbb{C}$ is the complex period of E which is defined to be the period integral $\Omega_E^+ = (2\pi\sqrt{-1}) \int_{E(\mathbb{R})} \omega_E$ associated to the Neron differential ω_E, $s(\phi)$ is the p-order of the conductor of ϕ and α is the p-unit root of $x^2 + a_p(E)x - p = 0$.

Later, several others tried to understand how generally the analytic p-adic L-function for M exists and how it should be characterized. The existence of the analytic p-adic L-function for M is hopeless if a motive M does not have any critical twists (Such situation happens when the motive M corresponds to the Artin representation for a certain weight-one cuspform). The conjecture on the existence and the characterization of the analytic p-adic L-function for motives M with critical twists was well-established in [CP].

2. A Generalization from a view point of Galois deformations

In the examples of the previous section, p-adic L-functions is defined on

$$\begin{aligned} \mathcal{X} &= \{\text{continuous characters } \eta : \Gamma \longrightarrow \overline{\mathbb{Q}}_p^\times\} \\ &\cong U(1;1) \subset \overline{\mathbb{Q}}_p, \end{aligned}$$

where $U(a,r) = \{x \in \overline{\mathbb{Q}}_p \mid |x-a|_p < r\}$ and the isomorphism is given by $\eta \mapsto \eta(\gamma)$ for a fixed topological generator $\gamma \in \Gamma$.

Let us reconsider the example of elliptic curves E. We associate the following family of Galois representations to an elliptic curve E over $\mathbb{Q}$:

$$\widetilde{T} := T_p(E) \otimes \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}),$$

where $T_p(E)$ is the p-Tate module of E, $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ is a rank-one free $\mathbb{Z}_p[[\Gamma]]$-module on which $G_{\mathbb{Q}}$ acts via the character $\widetilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathbb{Z}_p[[\Gamma]]$. We would like to remark the following two points:

(1) The specialization $\widetilde{T}_\phi$ of $\widetilde{T}$ at every character $\phi \in \mathcal{X}$ is isomorphic to the twisted Galois representation $T_p(E) \otimes \phi$. Hence we regard naturally $\widetilde{T}$ to be a family of rank-two Galois representation over the space $\mathcal{X}$.

(2) We associate the Hasse-Weil L-function $L(T,s) = \prod_{l:\text{primes}} \dfrac{1}{P_l(x;T)|_{x=l^{-s}}}$ to geometric Galois representations T, where

$$P_l(x;T) = \begin{cases} \det(1 - \mathrm{Frob}_l x; T^{I_l}) & \text{if } l \neq p, \\ \det(1 - \varphi x; D_{\mathrm{crys}}(T \otimes \mathbb{Q}_p)) & \text{if } l = p. \end{cases}$$

According to this definition, we have:

$$L(\widetilde{T}_\phi, s) = L(T_p(E) \otimes \phi, s) = L(E, \phi, s),$$

for each finite character ϕ, where $L(E,\phi,s)$ is the twist $\sum \dfrac{\phi(n)a_n(E)}{n^s}$ of the Hasse-Weil L-function $L(E,s) = \sum \dfrac{a_n(E)}{n^s}$.

The family $\widetilde{T}$ is called the cyclotomic deformation associated to $T_p(E)$. By the same procedure as above, we associate the cyclotomic deformation $T \otimes \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ to any p-adic representation T of $G_{\mathbb{Q}}$.

Thus, the situation for Theorem 1.3 is translated into the situation where we are given a Galois deformation space over a certain rigid analytic base space. From this point of view, we could generalize the framework for analytic p-adic L-functions even forgetting the cyclotomic $\mathbb{Z}_p$-extension. The ultimate generalization should be given through the following notion of the geometric triple.

Definition 2.1. We call a triple $(\mathcal{B}, \mathcal{T}, P)$ satisfying the following properties *a geometric triple*:

(1) $\mathcal{B}$ is an irreducible p-adic rigid analytic space (Mostly, we think of a finite cover of d-dimensional unit ball).
(2) $\mathcal{T}$ is a family of Galois representation, which is a finitely generated $\mathcal{O}(\mathcal{B})$-module with continuous action of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, where $\mathcal{O}(\mathcal{B})$ is the ring of "($\mathbb{Z}_p$-valued) holomorphic functions on $\mathcal{B}$" such that $\mathcal{B} = \mathrm{Spf}(\mathcal{O}(\mathcal{B}))$.
(3) P is a dense subset of $\mathcal{B}$ such that the fiber $\mathcal{T}_x$ at $x \in P$ is the p-adic Galois representation associated to a certain critical motive M_x over $\mathbb{Q}$.

Remark 2.2. (1) When $\mathcal{B}$ is a d-dimensional unit ball, we have $\mathcal{O}(\mathcal{B}) \cong \mathbb{Z}_p[[T_1, \cdots, T_d]]$. Thus, if $\mathcal{B}$ is a finite cover of d-dimensional unit ball, $\mathcal{O}(\mathcal{B})$ would be isomorphic to a finite extension of $\mathbb{Z}_p[[T_1, \cdots, T_d]]$. We will also remark that $\mathbb{Z}_p[[T_1, \cdots, T_d]]$ is non-canonically isomorphic to the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p^d]]$.
(2) Note that a motive M is defined to be critical if the special values of Hasse-Weil L-functions $L(M, 0)$ and $L(M^*(1), 0)$ do not vanish, except the case where $s = 0$ is contained in the critical strip of the functional equation for $L(M, s)$, where $M^*(1)$ is the Kummer dual of M. It is conjectured by Deligne that $L(M, 0)/\Omega^+_{M,\infty}$ is an algebraic number, where $\Omega^+_{M,\infty}$ is a complex period for M. We refer the reader to the article [De2] for the definitions of the notion of "critical", the definition of the complex period $\Omega^+_{M,\infty}$ and the conjecture of special values of the L-functions for critical motives.

A lot of such examples for $(\mathcal{B}, \mathcal{T}, P)$ are provided by Hida theory, Coleman theory and Mazur's theory on deformations of Galois representations.

Problem 2.3. Let $(\mathcal{B}, \mathcal{T}, P)$ be a geometric triple. Then, we would like to have a p-adic L-function $L_p(\mathcal{T})$ for $\mathcal{T}$, which is defined over $\mathcal{B}$ with the following interpolation property at each $x \in P$:

$$L_p(\mathcal{T})(x) = (\text{natural normalization factors at } x) \times \frac{L(M_x, 0)}{\Omega^+_{M_x,\infty}}$$

There are two important points related to this problem.

(1) We have to make clear the term "(natural normalization factors at x)" which is described, for example, by p-Euler factor of the L-function, Gauss

sum etc. We also have to well normalize the complex period $\Omega^+_{M_x}$. The complex period is defined to be the determinant of the isomorphism $H_{\mathrm{dR}}(M_x)^+ \otimes \mathbb{C} \cong H_{\mathrm{Betti}}(M_x)^+ \otimes \mathbb{C}$ with respect to the de Rham realization $H_{\mathrm{dR}}(M_x)^+$ and the Betti realization $H_{\mathrm{Betti}}(M_x)^+$. Since we have no canonical choice of these bases in general, we need to some how normalize the period $\Omega^+_{M_x}$.

(2) We have to make clear the ring where $L_p(\mathcal{T})$ is contained. If we allow arbitrary functions for $L_p(\mathcal{T})$, the problem of the construction of $L_p(\mathcal{T})$ is nonsense. In other words, we would like our p-adic L-function $L_p(\mathcal{T})$ to satisfy certain p-adic continuation property.

We tried to discuss these two points as explicitly as possible in the article [O6]. Hence we skip the precise description at the moment.

In the above two examples, $L_p(\mathcal{T})$ is an element of "the ring of bounded holomorphic functions" $\mathcal{O}(\mathcal{B})$ on which the Galois deformation is defined (Note that $\mathcal{O}(\mathcal{B}) = \mathbb{Z}_p[[\Gamma]]$ when $\mathcal{B} = \mathcal{X}$). However, in general it is not the case.

In order to give an example, let us introduce the subring $\mathcal{H}_r$ of $\mathbb{Q}_p[[T]]$ defined to be

$$\mathcal{H}_r := \{f(T) = \sum a_n(T-1)^n \mid \lim_{n\to\infty} |a_n| n^{-r} = 0\}.$$

We have the inclusion

$$\mathbb{Z}_p[[\Gamma]] \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \cdots \subset \mathcal{H}_r \subset \cdots ,$$

where $\mathbb{Z}_p[[\Gamma]]$ is embedded in $\mathcal{H}_1$ via the isomorphism $\mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \mathbb{Z}_p[[T]]$ which sends a topological generator γ of Γ to $1+T$.

Theorem 2.4 (Mazur-Tate-Teitelbaum)**.** *Let E be an elliptic curve which has good supersingular reduction at p. Fix a root α of $x^2 + a_p(E)x - p = 0$. Then there exists an element $L_p(E) \in \mathcal{H}_1$ with the following interpolation properties:*

$$\phi(L_p(E)) = \left(1 - \frac{\phi^{-1}(p)}{\alpha}\right)^2 \alpha^{-s(\phi)} G(\phi^{-1}) \frac{L(E, \phi^{-1}, 1)}{\Omega^+_E}.$$

Thus, the p-adic L-function $L_p(\mathcal{T})$ is not necessarily contained in $\mathcal{O}(\mathcal{B})$ even if $\mathcal{T}$ is a free of finite rank over $\mathcal{O}(\mathcal{B})$.

Several contributions are devoted to such generalizations of the p-adic L-functions before. We will recall three important ideas.

Review of the contributions.

(1) Greenberg proposed a generalization of Iwasawa main conjecture. He insists that the condition of "admissibility" (This is the property which he calls "Panchishkin condition" in [Gr3] which we do not make clear here) is important. He predicts the following statements when $\mathcal{T}$ is "admissible":
   (a) "The analytic p-adic L-function" $L_p(\mathcal{T})$ (if it exists and if it satisfies certain reasonable interpolation properties) is an element in $\mathcal{O}(\mathcal{B})$.
   (b) According to the admissibility of $\mathcal{T}$, we defines the Selmer group $\mathrm{Sel}_{\mathcal{T}}$ as a subgroup of $H^1(\mathbb{Q}, \mathcal{A})$ satisfying certain local condition where $\mathcal{A} = \mathcal{T} \otimes \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}(\mathcal{B}), \mathbb{Q}_p/\mathbb{Z}_p)$. The Pontrjagin dual of $\mathrm{Sel}_{\mathcal{T}}$ is a

finitely generated torsion $\mathcal{O}(\mathcal{B})$-module, whose characteristic ideal is equal to the ideal $(L_p(\mathcal{T})) \subset \mathcal{O}(\mathcal{B})$.

(2) Hida, who was influenced by work of Blasius, insisted the importance of the p-adic periods as well as complex periods in the interpolation property for the p-adic L-function $L_p(\mathcal{T})$. Briefly the idea is that we modify the p-adic interpolation as follows:

$$\frac{L_p(\mathcal{T})(x)}{\Omega^+_{M_x,p}} = (\text{natural normalization factors at } x) \times \frac{L(M_x,0)}{\Omega^+_{M_x,\infty}}.$$

If the p-adic period $\Omega^+_{M_x,p}$ is well-defined and we consider $L_p(\mathcal{T})(x)$ to be the one which satisfies the above interpolation, the ambiguity of $\Omega^+_{M_x,\infty}$ is canceled by the ambiguity of $\Omega^+_{M_x,p}$. This idea is shown in his book [H4].

(3) The above two are concerned only on "admissible" cases, Panchishkin explains in which extension of $\mathcal{O}(\mathcal{B})$ the analytic p-adic L-function $L_p(\mathcal{T})$ should be contained when $\mathcal{T}$ is not necessarily "admissible". For this purpose, he introduced the invariant which measures the difference of the Newton polygon and Hodge polygon. In [P1], the property of $L_p(\mathcal{T})$ is well-explained when $\mathcal{T}$ is a cyclotomic deformation which not "admissible".

In spite of progress stated above, still the picture for the p-adic L-functions $\mathcal{T}$ is not totally understood. In [O6], we tried to give a conjectural framework which overcome certain difficulties in general situations. For example, by slightly modifying the p-adic periods $\Omega^+_{M_x,p}$ we give a detailed description of the normalization factor in the interpolation property without abbreviating any terms. No convincing examples of $L_p(\mathcal{T})$ satisfying all such problems have been known except when $(\mathcal{B},\mathcal{T},P)$ are cyclotomic deformations. In the cases of Hida's nearly ordinary deformations, we give a detailed discussion to offer an example which will enlighten the general framework. This is also given in [O6], but we try to add explanations in the next section.

3. Hida deformations

Let $\mathcal{X}$ be an open unit ball which is the space of continuous characters on Γ. We define Γ' to be the p-Sylow subgroup of the group of diamond operators on the tower of modular curves $\{Y_1(p^n)\}_{n\geq 1}$. Recall that the diamond operator $\langle d\rangle$ ($d \in (\mathbb{Z}/p^n\mathbb{Z})^\times$) on $\{Y_1(p^n)\}_{n\geq 1}$ is the automorphism on which sends a classifying objects (E,e) with E an elliptic curve and e a p^n-torsion points on E to the classifying objects (E,de). Naturally Γ' is isomorphic to the p-Sylow subgroup $1+p\mathbb{Z}_p$ of $\mathbb{Z}_p^\times$. We have another space $\mathcal{Y}$ as follows:

$$\begin{aligned}\mathcal{Y} &= \{\text{continuous characters } \eta : \Gamma' \longrightarrow \overline{\mathbb{Q}}_p^\times\} \\ &\cong U(1;1) \subset \overline{\mathbb{Q}}_p,\end{aligned}$$

We have p-adic families $\mathcal{T}$ of Galois representations on a finite cover $\mathcal{B}_\mathcal{T}$ over $\mathcal{X}\times\mathcal{Y}$ called Hida's nearly ordinary deformations. The Hida deformations $\mathcal{T}$ have the following property:

(1) $\mathcal{T}$ is generically of rank two. Most of the time, $\mathcal{T}$ is free of rank two over $\mathcal{B}_{\mathcal{T}}$.
(2) For every arithmetic point $x \in \mathcal{B}_{\mathcal{T}}$ of weight $(j, k-2)$ with $k \geq 2$, $\mathcal{T}_x$ is isomorphic to the j-th Tate twist of the p-adic Galois representation associated to an eigen cuspform f_x of weight k.

This provides us the first interesting example of an "admissible" geometric triple $(\mathcal{B}_{\mathcal{T}}, \mathcal{T}, P)$ which is not cyclotomic. There have been several results related to p-adic L-function $L_p(\mathcal{T})$. We list known results on the construction for $L_p(\mathcal{T})$:

	method of construction	other remarks
Katz	Eisenstein measure	only CM cases
Yager	Coleman theory of ellip. units	only CM cases
Greenberg-Stevens [GS]	Λ-adic modular symbols	not necessarily optimally normalized
Kitagawa [Ki]	Λ-adic modular symbols	periods are optimally normalized
Fukaya [Fu]	Coleman theory for K_2	not necessarily optimally normalized
Ochiai [O3]	Coleman–Perrin-Riou theory	not necessarily optimally normalized
Panchishkin [P2]	Eisenstain family	not necessarily optimally normalized

The first two results by Katz and Yager are based on the theory of complex multiplications and they are constructions only for Hida deformations with complex multiplication. Greenberg-Stevens and Kitagawa gave constructions which are basically applied to every Hida deformations not necessarily of CM type and their method are generalizations of modular symbol method. However, in the paper by Greenberg-Stevens, efforts are payed on rather applications to the trivial zero conjecture of Mazur-Tate-Teitelbaum and the complex periods and p-adic periods are not well-optimized in general, as they point out by themselves. Hence the construction of [GS] does not match well with the BSD conjecture.

The results by Fukaya and Ochiai are based on Beilinson-Kato elements constructed by Kato. Panchishkin's method is to take a Rankin-Selberg integral of a family of Eisenstein series over $\mathcal{B}_{\mathcal{T}}$ and a family of modular forms $\{f_x\}_{x \in \mathcal{B}_{\mathcal{T}}}$. The construction by Fukaya, Ochiai and Panchishkin have problems that complex periods via Rankin-Selberg method in their interpolation terms were not normalized to give optimal periods.

Kitagawa's construction $L_p^{\mathrm{Ki}}(\mathcal{T})$, which is rather desirable, satisfies the following properties:

$$
\begin{aligned}
&(L_p^{\mathrm{Ki}}(\mathcal{T}))(x)/C_{p,x} \\
&= (-1)^{j-1}(j-1)! \left(1 - \frac{(\omega^{-j}\phi)(p)p^{j-1}}{a_p(f_x)}\right) \left(\frac{p^{j-1}}{a_p(f_x)}\right)^{s(j,\phi)} G(\omega^j\phi^{-1}) \frac{L(f_x, \omega^{-j}\phi, j)}{(2\pi\sqrt{-1})^{j-1}\Omega^{+}_{\infty,x}},
\end{aligned}
$$

where $s(j, \eta)$ is the p-order of the conductor of $\omega^j\phi^{-1}$ and $G(\omega^j\phi^{-1})$ is the Gauss sum for $\omega^j\phi^{-1}$, where

(1) $L_p^{\mathrm{Ki}}(\mathcal{T})$ is an element in $\mathcal{O}(\mathcal{B}_{\mathcal{T}})$.
(2) An error term $C_{p,x}$ is a p-adic unit in $\overline{\mathbb{Q}}_p$ at each x.

(3) $\Omega^+_{\infty,x}$ is a complex period associated to f_x which is optimized by choosing integral basis of the module of modular symbols and the specialization at every arithmetic point matches well with the BSD conjecture.

In [O3], we obtained a generalized Perrin-Riou map $\Xi : H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1)) \longrightarrow \mathcal{O}(\mathcal{B})$ which interpolates dual exponential $H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1)_x) \longrightarrow \overline{\mathbb{Q}}_p$ at $x \in \mathcal{B}_{\mathcal{T}}$. We defined the p-adic L-function to be an element $\Xi(\mathcal{Z}) \in \mathcal{O}(\mathcal{B})$ where $\mathcal{Z} \in H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1))$ is a natural Beilinson-Kato element. We modified our p-adic L-function by comparing with Kitagawa's construction:

Theorem 3.1. [O5] *There exists a modified Euler system $\mathcal{Z}^{\mathrm{Ki}} \in H^1(\mathbb{Q}_p, \mathcal{T}^*(1))$ such that $\Xi(\mathcal{Z}^{\mathrm{Ki}}) = L_p^{\mathrm{Ki}}(\mathcal{T})$.*

It seems possible to multiply an invertible element $U \in \mathcal{O}(\mathcal{B}) \otimes \mathcal{O}_{\mathbb{C}_p}$ with Kitagawa's two-variable p-adic L-function $L_p^{\mathrm{Ki}}(\mathcal{T}) \in \mathcal{O}(\mathcal{B})$ such that $L_p^{\mathrm{Ki}}(\mathcal{T}) \cdot U \in \mathcal{O}(\mathcal{B}) \otimes \mathcal{O}_{\mathbb{C}_p}$ has the interpolation property:

$$\begin{aligned}&\left(L_p^{\mathrm{Ki}}(\mathcal{T})(x) \cdot U(x)\right) / \Omega^+_{p,x} \\ &= (-1)^{j-1}(j-1)! \left(1 - \frac{(\omega^{-j}\phi)(p)p^{j-1}}{a_p(f_x)}\right) \left(\frac{p^{j-1}}{a_p(f_x)}\right)^{s(j,\phi)} G(\omega^j \phi^{-1}) \frac{L(f_x, \omega^{-j}\phi, j)}{(2\pi\sqrt{-1})^{j-1}\Omega^+_{\infty,x}},\end{aligned}$$

where $\Omega^+_{p,x}$ is a p-adic period defined by the comparison theorem of p-adic Hodge theory. In this way, $L_p^{\mathrm{Ki}}(\mathcal{T})$ is a convincing candidate for the analytic p-adic L-function used in the Iwasawa Main conjecture. On the other hand, we have studied Galois cohomologies and Selmer groups for Hida deformations and we established the candidate $L_p^{\mathrm{alg}}(\mathcal{T})$ for the algebraic p-adic L-function used in the Iwasawa Main Conjecture to be the characteristic ideal of the Selmer group for $\mathcal{T}$.

Iwasawa Main Conjecture for $(\mathcal{B}, \mathcal{T}, P)$. We have an equality of ideal

$$(L_p^{\mathrm{Ki}}(\mathcal{T})) = (L_p^{\mathrm{alg}}(\mathcal{T})).$$

This type of conjecture was first proposed by Greenberg in early 90's as a rough sketch without specifying which p-adic L-functions to choose and without other various observations. Hence we revisited the conjecture adding convincing observations and modifications. Further, by our Euler system theory, we have established the following inequality:

Theorem 3.2. *We have an inequality:*

$$(L_p^{\mathrm{Ki}}(\mathcal{T})) \subset (L_p^{\mathrm{alg}}(\mathcal{T})).$$

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