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ON POWER INTEGRAL BASES OF THE 2-ELEMENTARY ABELIAN EXTENSION FIELDS

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Abstract. Let K be an abelian field whose Galois group is 2-elementary abelian over the rationals $\boldsymbol{Q}$. If K is monogenic and it is generated by a quadratic subfield and a quartic subfield which are linearly disjoint, then K coincides with the field $\boldsymbol{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{-3})$, namely K is equal to the cyclotomic field $\boldsymbol{Q}(\zeta_{24})$ [MN]. In this article, we prove that all the real and imaginary octic fields K are non-monogenic, namely the rings Z_K of integers in K do not have any power integral basis except for $\boldsymbol{Q}(\zeta_{24})$. Our method includes succinct new proofs for the linearly disjoint case of [MN] and for Proposition 2 [PNM] as a main tool of our purpose.

1. Introduction

Let K be an algebraic number field of extension degree $[K : \boldsymbol{Q}] = n$ over the rationals $\boldsymbol{Q}$. We denote the ring of integers in K by Z_K. When $Z_K = \boldsymbol{Z}[\alpha] = \boldsymbol{Z}[1, \alpha, \cdots, \alpha^{n-1}]$ for some element α of Z_K, it is said that α generates a power integral basis of the ring Z_K or simply Z_K has a power integral basis. The field K is called monogenic if Z_K has a power integral basis. It is known as a problem of Hasse to characterize whether a field K is monogenic or not[Gy]. In this article, we consider the fields K whose Galois groups are 2-elementary abelian. If $[K : \boldsymbol{Q}] \geq 16$, then K is non-monogenic, i.e., the ring Z_K of integers in K has no power integral basis by virtue of the decomposition theory of a prime number (Lemma 1, [SN], [MNS], [Wa]). By the work [Wi], [GT] of K. S. Williams, M.-N. Gras and F. Tanoé for Dirichlet fields K, it is enough for us to investigate the field $K = \boldsymbol{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1 m_1 n_1 \ell})$, where $d = d_1 d_2, m = m_1 m_2, n = n_1 n_2, mn \equiv 3, dn \equiv 2, d_1 m_1 n_1 \ell \equiv 1, d_2 \equiv 2 \pmod 4), d_1, m_1, n_1 \geq 1$ and $dmnl$ is square free. Let k and L be a quadratic subfield $\boldsymbol{Q}(\sqrt{d_1 m_1 n_1 \ell})$ and a quartic subfield $\boldsymbol{Q}(\sqrt{mn}, \sqrt{dn})$ of K, respectively. Then in the case of $d_1 m_1 n_1 = 1$, namely k and L are linearly disjoint, such an octic field $K = kL$ is non-monogenic except for the cyclotomic field $\boldsymbol{Q}(\zeta_{24})$ of conductor 24 [MN]. In this paper, we will use the relative discriminant $D_{K/L}$ for any Dirichlet subfield L in an octic field K [Lemma 2]. Next, being based

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on the identity

$$(\xi - \xi^{\sigma})(\xi - \xi^{\sigma})^{\rho} - (\xi - \xi^{\rho})(\xi - \xi^{\rho})^{\sigma} + (\xi - \xi^{\sigma\rho})(\xi - \xi^{\sigma\rho})^{\sigma} = 0,$$

namely, on the linear equation for the valuables $\ell, 2d_2, d_1$,

$$\ell E_{11} - 2d_2 E_{12} + d_1 E_{13} = 0$$

for a primitive element ξ in K, automorphisms

$$\sigma : \sqrt{dn} \mapsto -\sqrt{dn}, \rho : \sqrt{d_1 m_1 n_1 \ell} \mapsto -\sqrt{d_1 m_1 n_1 \ell}$$

of $K/\boldsymbol{Q}$ and units E_{ij} in the fixed field $k_1 = \boldsymbol{Q}(\sqrt{D_1}), D_1 = 4mn = m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2$, together with the other six similar equations [Proposition 2], we will prove that all the octic 2-elementary abelian fields K are non-monogenic except for $\boldsymbol{Q}(\zeta_{24})$ [Theorem]. Then we will claim that all the real 2-elementary abelian fields of degree 8 have no power integral basis.

2. The relative different

We determine the relative different $D_{K/L}$ of any Dirichelet field L in an octic field K whose Galois group is 2-elementary abelian. We denote the Galois group $\langle \tau, \sigma, \rho \quad | \quad \tau : \sqrt{mn} \mapsto -\sqrt{mn}, \sigma : \sqrt{dn} \mapsto -\sqrt{dn}, \rho : \sqrt{d_1 m_1 n_1 \ell} \mapsto -\sqrt{d_1 m_1 n_1 \ell} \rangle$ of $K/\boldsymbol{Q}$ by G with the identity I.

The following lemma and proposition are available to deduce the type of 2-elementary abelian extension fields K which have power integral bases.

Lemma 1([SN]). *Let ℓ be a prime number and let $F/\boldsymbol{Q}$ be a Galois extension of degree $n = efg$ with ramification index e and the relative degree f with respect to ℓ. If one of the following conditions is satisfied, then Z_F has no power integral basis, i.e., F is non-monogenic ;*
(1) $e\ell^f < n$ *if* $f = 1$;
or
(2) $e\ell^f \leq n + e - 1$ *if* $f \geq 2$.

Proposition 1([MN]). *Let $a_1, a_2, \cdots, a_r$ be square free rational integers and F be the field $\boldsymbol{Q}(\sqrt{a_1}, \sqrt{a_2}, \cdots, \sqrt{a_r})$ of degree $2^r, r \geq 4$. Then F is non-monogenic.*

Proof. Without loss of generality, we may assume that there exists at most two generators $\sqrt{a_1}, \sqrt{a_2}$ of F with $a_j \not\equiv 1 \pmod 4 (1 \leq j \leq 2)$. Then the ramification index e of the prime is at most 2^2. Since the Galois group $G = Gal(F/\boldsymbol{Q})$ is 2-elementary, the relative degree f of the prime 2 is at most 2, because the inertia subgroup of G is cyclic. Let ℓ be equal to 2 in Lemma 1. Then we can deduce $e\ell^f \leq 2^2 \cdot 2^1 < 2^r$ if $f = 1$ and $e\ell^f \leq 2^2 \cdot 2^2 \leq 2^r + e - 1$ if $f = 2$. Thus F is non-monogenic. ∎

For algebraic number fields $M \supset N \supset \boldsymbol{Q}$, let D_M and $D_{M/N}$ be the field discriminant of M and the relative discriminant of M/N, respectively. Our theorem is based on the following lemma.

LEMMA 2. *Let L_γ be the Dirichlet field fixed by a subgroup $<\gamma>\neq<I>$ of the Galois group $G(K/\boldsymbol{Q})$ for the octic field $K=\boldsymbol{Q}(\sqrt{mn},\sqrt{dn},\sqrt{d_1m_1n_1\ell})$ with $d=d_1d_2, m=m_1m_2,\ n=n_1n_2, mn\equiv 3, dn\equiv 2, d_1m_1n_1\ell\equiv 1, d_2\equiv 2 \pmod 4), d_1,m_1,n_1\geq 1$ and $dmn\ell$ is square free. Then we have for $\ell_0=d_1m_1n_1$*

$$D_K=2^{12}(dmn\ell)^4$$

and

$$\begin{aligned}
L_\tau&=\boldsymbol{Q}(\sqrt{dn},\sqrt{\ell_0\ell}) & D_{L_\tau}&=2^4(dm_1n\ell)^2 & D_{K/L_\tau}&\cong 2m_2,\\
L_\sigma&=\boldsymbol{Q}(\sqrt{mn},\sqrt{\ell_0\ell}) & D_{L_\sigma}&=2^4(d_1mn\ell)^2 & D_{K/L_\sigma}&\cong 2d_2,\\
L_\rho&=\boldsymbol{Q}(\sqrt{mn},\sqrt{dn}) & D_{L_\rho}&=2^6(dmn)^2 & D_{K/L_\rho}&\cong \ell,\\
L_{\tau\sigma}&=\boldsymbol{Q}(\sqrt{dm},\sqrt{\ell_0\ell}) & D_{L_{\tau\sigma}}&=2^4(dmn_1\ell)^2 & D_{K/L_{\tau\sigma}}&\cong 2n_2,\\
L_{\tau\rho}&=\boldsymbol{Q}(\sqrt{mn\ell_0\ell},\sqrt{dn}) & D_{L_{\tau\rho}}&=2^6(m_2nd\ell)^2 & D_{K/L_{\tau\rho}}&\cong m_1,\\
L_{\sigma\rho}&=\boldsymbol{Q}(\sqrt{mn},\sqrt{dn\ell_0\ell}) & D_{L_{\sigma\rho}}&=2^6(d_2mn\ell)^2 & D_{K/L_{\sigma\rho}}&\cong d_1,\\
L_{\tau\sigma\rho}&=\boldsymbol{Q}(\sqrt{dm},\sqrt{mn\ell_0\ell}) & D_{L_{\tau\sigma\rho}}&=2^6(dmn_2\ell)^2 & D_{K/L_{\tau\sigma\rho}}&\cong n_1,
\end{aligned}$$

where $\alpha\cong\beta$ means that (α) and (β) are equal to each other as ideals.

Proof. Let $\mathfrak{d}_M$ and $\mathfrak{d}_{K/L_\sigma}$ be the different of a field $M/\boldsymbol{Q}$ and the relative different $(\alpha-\alpha^\sigma;\alpha\in Z_K)$ with respect to K/L_σ. By virtue of the transitive law of the differents

$$\mathfrak{d}_K=\mathfrak{d}_{K/L_\sigma}\mathfrak{d}_{L_\sigma},$$

we have

$$N_K(\mathfrak{d}_K)=N_{L_\sigma}(N_{K/L_\sigma}(\mathfrak{d}_{K/L_\sigma}))N_{K/L_\sigma}(N_{L_\sigma}(\mathfrak{d}_{L_\sigma})),$$

namely $D_K\cong D^4_{K/L_\sigma}\cdot D^2_{L_\sigma}$, where $N_M(\mathfrak{a})$ means the norm of an ideal $\mathfrak{a}$ with respect to $M/\boldsymbol{Q}$ [Wa]. Since

$$D_{L_\sigma}=\pm N_{L_\sigma}(\mathfrak{d}_{L_\sigma})=\pm\prod_{\chi\in X_{L_\sigma}}f_\chi,$$

where X_{L_σ} is the character group corresponding to the Dirichlet field L_σ, we obtain $2d_2\cong D_{K/L_\sigma}$. In the same way we have the relative discriminants for the other six cases. ∎

3. THE CASE OF $d_1m_1n_1\geq 1$

It is known that in the case of $d_1m_1n_1=1$ that is, there exist a quartic subfield L and a quadratic k of K with $(D_L,D_k)=1$, the fields K are non-monogenic except for the cyclotomic field $\boldsymbol{Q}(\zeta_{24})$ of conductor 24 [MN]. From now on, we consider the case of $d_1m_1n_1\geq 1$ and as an application of Lemma 2, we can slightly generalize

Proposition 5 in [MN] whose proof was done using the relative different $\mathfrak{d}_{K/L}$ with respect to K over a suitable quartic subfield L. We assume that K is *monogenic.*

Then we have

$$d_K(\xi) = \pm N_K\big(\mathfrak{d}(\xi)\big) = \pm D_K,$$

where $d_K(\alpha), \mathfrak{d}(\alpha)$ and $N_K(\alpha)$ mean the discriminant, the different and the norm of a number α with respect to $K/\boldsymbol{Q}$, respectively. Then, by Lemma 2 for $(\xi-\xi^\sigma)(\xi-\xi^\sigma)^\rho = \eta_{11}$,
$(\xi-\xi^\rho)(\xi-\xi^\rho)^\sigma = \eta_{12}$ and $(\xi-\xi^{\sigma\rho})(\xi-\xi^{\sigma\rho})^\sigma = \eta_{13}$, we have $\eta_{11} = 2d_2E_1$, $\eta_{12} = \ell E_2$, and $\eta_{13} = d_1E_3$. Thus, because η_{1j} is a partial factor of $d_{K/Q}(\xi)$, integers E_j should be units in $k_1 = \boldsymbol{Q}(\sqrt{mn})$. Then by the following basic identity:

$$(\xi-\xi^\sigma)(\xi-\xi^\sigma)^\rho - (\xi-\xi^\rho)(\xi-\xi^\rho)^\sigma + (\xi-\xi^{\sigma\rho})(\xi-\xi^{\sigma\rho})^\sigma = 0$$

we have the equation

$$2d_2E_1 - \ell E_2 + d_1E_3 = 0$$

where E_1, E_2 and E_3 are units in $k_1 = \boldsymbol{Q}(\sqrt{m_1m_2n_1n_2})$.

In the same way, we obtain the other six equations corresponding to each of the six quadratic subfields k_j of K.

PROPOSITION 2. *If $K = \boldsymbol{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1m_1n_1\ell}\,)$ is monogenic, then the following simultaneous equations hold:*

(1) $\ell E_{11}+2d_2E_{12}+d_1E_{13} = 0$ *in* $k_1 = \boldsymbol{Q}(\sqrt{D_1})$, $D_1 = m_1\cdot 2m_2\cdot n_1\cdot 2n_2$,

(2) $\ell E_{21}+2m_2E_{22}+m_1E_{23} = 0$ *in* $k_2 = \boldsymbol{Q}(\sqrt{D_2})$, $D_2 = d_1\cdot 2d_2\cdot n_1\cdot 2n_2$,

(3) $\ell E_{31}+2n_2E_{32}+n_1E_{33} = 0$ *in* $k_3 = \boldsymbol{Q}(\sqrt{D_3})$, $D_3 = d_1\cdot 2d_2\cdot m_1\cdot 2m_2$,

(4) $2d_2E_{41}+2m_2E_{42}+2n_2E_{43} = 0$ *in* $k_4 = \boldsymbol{Q}(\sqrt{D_4})$, $D_4 = d_1\cdot m_1\cdot n_1\cdot \ell$,

(5) $2d_2E_{51}+m_1E_{52}+n_1E_{53} = 0$ *in* $k_5 = \boldsymbol{Q}(\sqrt{D_5})$, $D_5 = d_1\cdot 2m_2\cdot 2n_2\cdot \ell$,

(6) $d_1E_{61}+2m_2E_{62}+n_1E_{63} = 0$ *in* $k_6 = \boldsymbol{Q}(\sqrt{D_6})$, $D_6 = 2d_2\cdot m_1\cdot 2n_2\cdot \ell$,

(7) $d_1E_{71}+m_1E_{72}+2n_2E_{73} = 0$ *in* $k_7 = \boldsymbol{Q}(\sqrt{D_7})$, $D_7 = 2d_2\cdot 2m_2\cdot n_1\cdot \ell$,

where each E_{ij} is a unit in the corresponding quadratic subfield k_i of K and each D_i the field discriminant of k_i, respectively.

For the case of a real quadratic field, the following lemma holds:

LEMMA 3[MN]. *Let E_j be a power ${\varepsilon_0}^j = \frac{u_j+v_j\sqrt{D}}{2}$ of the fundamental unit $\varepsilon_0 = \frac{u+v\sqrt{D}}{2} > 1$ in a real quadratic field $\boldsymbol{Q}(\sqrt{D})$ with the field discriminant D and*

$\overline{\alpha} = \alpha^{\gamma}$ *for* α *in* $\boldsymbol{Q}(\sqrt{D})$ *and* $\gamma(\neq I)$ *in* $Gal\big(\boldsymbol{Q}(\sqrt{D})/\boldsymbol{Q}\big)$. *Let*

$$(*) \qquad \begin{cases} a + bE_j + cE_k = 0, \\ a + b\overline{E_j} + c\overline{E_k} = 0 \end{cases}$$

for $abc \neq 0$. *Denote the matrix*

$$\begin{pmatrix} 1 & E_j & E_k \\ 1 & \overline{E_j} & \overline{E_k} \end{pmatrix}$$

attached to the the equation $(*)$ *by* A *and the rank of* A *by* r_D. *Then we have a solution* (a, b, c) *of rational integers* :

$$\begin{cases} a \pm b \pm c = 0 \qquad for \qquad r_D = 1, \\ \dfrac{a}{u_k v_j - u_j v_k} = \dfrac{b}{2v_k} = \dfrac{c}{-2v_j} \qquad for \qquad r_D = 2 \end{cases}$$

with $E_i = \dfrac{u_i + v_i\sqrt{D}}{2}$.

In the case of *any* octic field $\boldsymbol{Q}(\sqrt{m_1 m_2 n_1 n_2}, \sqrt{d_1 d_2 n_1 n_2}, \sqrt{d_1 m_1 n_1 \ell})$, by the following lemma, we can deduce to evaluate the rank r_D of a quadratic field $\boldsymbol{Q}(\sqrt{D})$ for a few cases with respect to the order of values $d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell$ in the set of seven parameters.

Lemma 4[PNM]. *Let denote the set* $\{d_1, 2d_2, m_1, 2m_2, n_1, 2n_2, \ell\}$ *by* D. *Then it holds that:*

(1) *For one parameter* s *in* D, *there exist only four quadratic subfields* k_j *whose discriminants* D_j *are divisible by* s.

(2) *For two parameters* s, t *in* D, *there exist only two quadratic subfields* k_j *whose discriminants* D_j *are divisible by* st.

(3) *Let* s, t, u *be three parameters in* D, *such that* stu *is a divisor of the field discriminant of* D_j *of* k_j. *Then there exists only one quadratic subfield* k_j *whose discriminant* D_j *is divisible by* stu.

Remark 1. *We can confirm that the number of triplets* (s, t, u) *within the order of parameters in* D *is equal to* $28 = 7 \times \binom{4}{3} < \binom{\sharp D}{3} = 35$ *such that each of* stu *is a divisor of the field discriminant* D_j *of* k_j.

Next, we prepare the key lemma for the proof of the theorem.

Lemma 5[PNM]. *For the set* $D = \{a, b, c, d, e, f, g\}$ *of seven positive rational integers, assume that* $a > b \geq c > \max\{d, e, f, g\}$ *and* $d > f$ *or* $a > b > c \geq \max\{d, e, f, g\}$ *and* $d > f$. *Then*
(1) *For the field* $\boldsymbol{Q}(\sqrt{bcst})$, *where* $s, t \in D \backslash \{a, b, c\}$ *and units* E_i *in* $\boldsymbol{Q}(\sqrt{bcst})$, *the*

rank r_{bcst} *of the equations*

$$\begin{cases} a + uE_j + vE_k = 0, \\ a + u\overline{E_j} + v\overline{E_k} = 0, \end{cases}$$

with $\{u, v\} = D \backslash \{a, b, c, s, t\}$ *is equal to* 1.
(2) *For the field* $\boldsymbol{Q}(\sqrt{astu})$, *where* $s, t, u \in D \backslash \{a, b, c\}$ *and units* E_i *in* $\boldsymbol{Q}(\sqrt{astu})$, *the rank* r_{astu} *of the equations*

$$\begin{cases} b + cE_j + vE_k = 0, \\ b + c\overline{E_j} + v\overline{E_k} = 0, \end{cases}$$

with $\{v\} = D \backslash \{a, b, c, s, t, u\}$ *is equal to* 1.

Finally, we prove the following main theorem.

THEOREM. *Let* $K = \boldsymbol{Q}(\sqrt{a_1}, \cdots, \sqrt{a_r})$ *be a 2-elementary abelian extension field over* $\boldsymbol{Q}$ *whose degree* 2^r *is not less than* 8 *for square free integers* $a_1, \cdots, a_r$. *Then the field* K *is monogenic if and only if* $K = \boldsymbol{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{-3}) = \boldsymbol{Q}(\zeta_{24})$.

Proof. By Proposition 1, it is enough to consider an octic field K. Let $(2) = \mathfrak{L}_1{}^e \cdots \mathfrak{L}_g{}^e$ be the prime ideal decomposition of a rational prime 2 in K. For the ramification index e of 2, if $e \leq 1$, then by Lemma 1 and the relative degree f of a prime 2 is at most 2, we have $1 \cdot 2^1 < 8$ or $1 \cdot 2^2 \leq 8 + 1 - 1$ for $e = 1$ and $2 \cdot 2^1 \leq 8$ or $2 \cdot 2^2 \leq 8 + 2 - 1$ for $e = 2$, namely K is non-monogenic. Then in the case of $e \geq 3$, we can deduce that the type of an octic field K is $K = \boldsymbol{Q}(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$, where $a_1 = mn \equiv 3, a_2 = dn \equiv 2, a_3 = d_1 m_1 n_1 \ell \equiv 1 \pmod 4$, for $d = d_1 d_2, m = m_1 m_2, n = n_1 n_2$ and $dmn\ell$ is square free. Put $D = \{d_1, 2d_2, m_1, 2m_2, n_1, 2m_2, \ell\}$. We denote *again* by $\{a, b, c, d, e, f, g\}$ any transposition on the seven parameters in D. Without loss of generality, we may assume that $a > b \geq c > \max\{d, e, f, g\}$ and $d > f$ or $a > b > c \geq \max\{d, e, f, g\}$ and $d > f$. Using Lemma 5, it is enough for us to consider the following four cases.

Case (I). The field K includes $k_{j_1} = \boldsymbol{Q}(\sqrt{abct})$ for some $t \in D \backslash \{a, b, c\}$, e.g., $t = d$.
Then there exists $k_{j_2} = \boldsymbol{Q}(\sqrt{bcs'g})$ for some $s' \in D \backslash \{b, c, g, a, d\} = \{e, f\}$ by Lemma 4 (3). Thus by Lemma 5 (1) we have for $s \in D \backslash \{b, c, g, a, d\} = \{e, f\}, s \neq s'$

$$(\mathrm{I})_2 \qquad a + dE_j + sE_k = 0, \quad \text{hence} \quad a = d + s.$$

Next, by $k_{j_1}, k_{j_2}, (abcd) \cdot (bcs'g) \sim ads'g$, we have the third field $k_{j_3} = \boldsymbol{Q}(\sqrt{ads'g})$. Then by Lemma 5 (2), we have

$$(\mathrm{I})_3 \qquad b + cE_j + sE_k = 0, \quad \text{hence} \quad b = c + s.$$

By $(\mathrm{I})_{2,3}$, $d > c$, which is a contradiction to $c \geq d$.

Case (II). The field K does not include the field $\boldsymbol{Q}(\sqrt{abcs})$ for any $s \in D \backslash \{a, b, c\}$.
First we choose the field $k_{j_1} = \boldsymbol{Q}(\sqrt{abeg})$. By Lemma 4 (3), we can select the field $k_{j_2} = \boldsymbol{Q}(\sqrt{abtf})$ for $t \in D \backslash \{a, b, f, e, g, c\} = \{d\}$. Next we consider the third

field k_{j_3}, which is not included in the quartic field $L = k_{j_1} \cdot k_{j_2} = \boldsymbol{Q}(\sqrt{abeg}, \sqrt{abdf})$. We can select $k_{j_3} = \boldsymbol{Q}(\sqrt{bcsg})$ for $s, s' \in D\backslash\{b, c, g, a, e\} = \{d, f\}$, $s \neq s'$. Then

$$(\mathrm{II})_3 \qquad a + eE_j + s'E_k = 0, \quad \text{hence} \quad a = e + s'.$$

Furthermore, by $k_{j_1}, k_{j_2}, (abeg)\cdot(abdf) \sim defg$, we have the fourth $k_{j_4} = \boldsymbol{Q}(\sqrt{defg})$ in L. By $k_{j_1}, k_{j_3}, (abeg)\cdot(bcsg) \sim acse$, we have the fifth field $k_{j_5} = \boldsymbol{Q}(\sqrt{acse})$. Hence

$$(\mathrm{II})_5 \qquad b = s' + g.$$

By $k_{j_2}, k_{j_3}, (abss') \cdot (bcsg) \sim acs'g$, we have the sixth field $k_{j_6} = \boldsymbol{Q}(\sqrt{acs'g})$. Hence

$$(\mathrm{II})_6 \qquad b = s + e.$$

Finally, by $k_{j_1}, k_{j_2}, k_{j_3}, (abeg) \cdot (abss') \cdot (bcsg) \sim bces'$, we have the seventh field $k_{j_7} = \boldsymbol{Q}(\sqrt{bces'})$. Hence

$$(\mathrm{II})_7 \qquad a = s + g.$$

We know that $e > g$ by $(\mathrm{II})_{3,5}$ and $s' > s$ by $(\mathrm{II})_{3,6}$. From $(\mathrm{II})_{3,7}$ we deduce that $a = e + s' = s + g$. This is a contradiction. Then any real octic fields K does not have a power integral basis.

Finally, we consider an imaginary octic field K. Here we notice that for any imaginary quadratic subfield k_i in K, Proposition 2 holds for $E_{ij} \in U_i$, where $U_i = \{\pm 1\}, \{\pm 1, \pm\sqrt{-1}\}$ and $\{\pm 1, \pm\omega, \pm\omega^2\}$ with $\omega = \dfrac{1+\sqrt{-3}}{2}$ for the field discriminant D_{k_i} of $k_i, D_{k_i} < -4$, $D_{k_i} = -4$ and $D_{k_i} = -3$, respectively. If K includes the Gaußian field $\boldsymbol{Q}\big(\sqrt{1 \cdot 2(-1) \cdot 1 \cdot 2(1)}\big) = \boldsymbol{Q}(\sqrt{m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2}) = \boldsymbol{Q}(\sqrt{e \cdot -|c| \cdot f \cdot d}) = k_1$, then for the rank r_{-4} of the corresponding equation

$$(\mathrm{III})_0 \qquad |a| + |b|E_j + |g|E_k = 0, \qquad E_i \in U_{k_1}$$

it is equal to 1. Otherwise if $r_{-4} > 1$, then $E_j = \pm\sqrt{-1}$ or $E_k = \pm\sqrt{-1}$, we have $b = 0$ and $g = 0$ or $b \pm g = 0$, where, since the number of even parameters is three, only one of b or g is even, because, if both of b and g are odd, then we can consider the equation $(\mathrm{III})_0$ dividing by E_j as a new one; this is a contradiction. Moreover if an imaginary quadratic subfield $k_j = \boldsymbol{Q}(\sqrt{D})$ in K is neither the Eisenstein field $\boldsymbol{Q}(\sqrt{-3})$ nor the Gaußian field $\boldsymbol{Q}(\sqrt{-1})$, the unit group U_{k_j} is $\{\pm 1\}$. Then the rank r for such an imaginary field, $r_D = 1$.

Case (III). Let K include the Eisenstein field

$$\boldsymbol{Q}(\sqrt{-3}) = \boldsymbol{Q}(\sqrt{1 \cdot 1 \cdot 1 \cdot (-3)}) = \boldsymbol{Q}(\sqrt{d_1 \cdot m_1 \cdot n_1 \cdot \ell}) = \boldsymbol{Q}(\sqrt{e \cdot f \cdot g \cdot -|b|}).$$

Then let the quadratic subfield k_4 coincides with $\boldsymbol{Q}(\sqrt{-3})$. Then any such octic field K should belong to the family of the linearly disjoint case in [MN]. Thus such an imaginary octic field K is non-monogenic except for

$$\boldsymbol{Q}(\sqrt{-4}, \sqrt{8}, \sqrt{-3}) = \boldsymbol{Q}(\sqrt{m_1 \cdot 2m_2 \cdot n_1 \cdot 2n_2}, \sqrt{d_1 \cdot 2d_2 \cdot n_1 \cdot 2n_2}, \sqrt{\ell_0 \ell}) = \boldsymbol{Q}(\zeta_{24})$$

of the cyclotomic field of 24-th root ζ_{24} of unity, where $a = 2d_2 = 2(2)$, $b = \ell = -3$, $c = 2m_2 = 2(-1)$, $d = e = f = g = 1$. We know that $Z_K = \boldsymbol{Z}[\zeta_{24}]$, that is, K has a power integral basis.

Case (IV). Let K do not include the Eisenstein field $\boldsymbol{Q}(\sqrt{-3})$. For any imaginary quadratic subfield k_j in K, the unit group U_{k_j} is $\{\pm 1\}$ or $\{\pm 1, \pm\sqrt{-1}\}$. Then the rank r_j of the corresponding equation of k_j is equal to 1. Then we obtain that such an imaginary octic field K is non-monogenic by virtue of the same consideration as in the real case (I) or case (II) with $|c| \geq |d| > |f|$. Therefore we have proved Theorem. □

4. Problem

By [PMN], it is known an explicit integral basis of some real octic field, which is an extension of a result of the case of quartic fields $[\mathrm{M}_1, \mathrm{M}_2, \mathrm{Wi}]$.

Theorem 1([PMN]). *Let K be an octic field $\boldsymbol{Q}(\sqrt{mn}, \sqrt{dn}, \sqrt{d_1 m_1 n_1 \ell})$ with $d = d_1 d_2, m = m_1 m_2, n = n_1 n_2, mn \equiv 3, dn \equiv 2, d_1 m_1 n_1 \ell \equiv 1, d_2 \equiv 2 \pmod 4), d_1, m_1, n_1 \geq 1$ and $dmn\ell$ is square free. Let D_K be the field discriminant of the octic field K. Then we have $D_K = 2^{12}(dmn\ell)^4$ and an integral basis of K is :*

$$Z_K = Z\Big[1, \sqrt{mn}, \sqrt{dn}, \frac{\sqrt{dm}+\sqrt{dn}}{2}, \frac{1+\sqrt{d_1 m_1 n_1 \ell}}{2}, \boldsymbol{Q}(\sqrt{d_1 m_1 n_1 \ell})\frac{\sqrt{mn}+\sqrt{d_1 m_2 n_2 \ell}}{2}, \frac{\sqrt{dn}+\sqrt{d_2 m_1 n_2 \ell}}{2}, \frac{\sqrt{dm}+\sqrt{dn}+e_1\sqrt{d_2 m_2 n_1 \ell}+e_2\sqrt{d_2 m_1 n_2 \ell}}{4}\Big]$$

where $e_i = \pm 1$ $(i = 1, 2), e_1 \equiv d_1 m_1, e_2 \equiv d_1 n_1 \pmod 4)$.

Concerning the above theorems, we propose the followings;

Problem. For a primitive elment ξ in K, let $\mathrm{Ind}(\xi)$, $\tilde{m}(K)$ and $m(K)$ be the index $\sqrt{\left|\frac{d_K(\xi)}{D_K}\right|}$ of an elemnet ξ, the minimum index $\min_{\xi\in K}\{\mathrm{Ind}(\xi)\}$ of K and the field index $\gcd_{\xi\in K}\{\mathrm{Ind}(\xi)\}$ of K, respectively. Let the fields K run through all the real octic fields whose Galois groups are 2-elementary abelian. Then evaluate the values of

$$\inf_K \tilde{m}(K) \quad \text{and} \quad \inf_K m(K),$$

respectively.

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