ON THE $K$-GROUPS OF ALGEBRAIC INTEGERS OF CYCLOTOMIC FIELDS

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Let $L$ be a number field, $\mathcal{O}_L$ be the ring of algebraic integers of $L$. For any integer $n \geq 0$, we denote the $n$-th Quillen $K$-group of $\mathcal{O}_L$ by $K_n(\mathcal{O}_L)$, especially we have $K_0(\mathcal{O}_L) \simeq \mathbb{Z} \oplus \text{Pic}(\mathcal{O}_L)$ and $K_1(\mathcal{O}_L) \simeq \mathcal{O}_L^\times$. The following facts on the $K$-groups are well known.

**Theorem 1.** (Quillen[14]) For any integer $n \geq 0$, $K_n(\mathcal{O}_L)$ is a finitely generated abelian group.

**Theorem 2.** (Borel [2]) For any integer $n \geq 0$, we have

$$\text{rank}_2 K_n(\mathcal{O}_L) = \begin{cases} 0 & \text{if } n \geq 2 \text{ and even}, \\ r_1 + r_2 & \text{if } n \geq 5 \text{ and } n \equiv 1 \pmod{4}, \\ r_2 & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $r_1$ or $r_2$ is the number of the real or complex places of $L$.

**Example 1.** The structures of the $K$-groups of $\mathbb{Z}$ for lower degree are known as follows. $K_0(\mathbb{Z}) \simeq \mathbb{Z}$, $K_1(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, $K_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ (Milnor [11]), $K_3(\mathbb{Z}) \simeq \mathbb{Z}/48\mathbb{Z}$ (Lee-Szczarba [9]), $K_4(\mathbb{Z}) = 0$ (Rognes [15]), $K_5(\mathbb{Z}) \simeq \mathbb{Z}$ (Elbaz Vincent-Gangl-Soulé [4]).

Kurihara [8] and Mitchell [12] conjectured independently the following statement.

**Conjecture 1.** (Kurihara [8,Conjecture 3.2], Mitchell [12, 6.15]) For any integer $n \geq 0$, we have

- $K_{4n}(\mathbb{Z}) \simeq' 0 \quad (n \geq 1)$
- $K_{4n+1}(\mathbb{Z}) \simeq' \mathbb{Z} \quad (n \geq 1)$
- $K_{4n+2}(\mathbb{Z}) \simeq' \mathbb{Z}/N_{2n+2}\mathbb{Z}$
- $K_{4n+3}(\mathbb{Z}) \simeq' \mathbb{Z}/D_{2n+2}\mathbb{Z}$
where for an even integer \(k\), the positive integers \(N_k\) and \(D_k\) are given by \(\zeta(1-k) = -\frac{B_k}{k}\) 

\[ (-1)^{k/2} \frac{N_k}{D_k} \] with \((N_k, D_k) = 1\). The Bernoulli number \(B_k\) is the same notation as in [8] or [18], and ‘\(\simeq\)’ means an isomorphism up to 2-torsion groups.

Note that Mitchell [12] give the statement including 2-torsion groups. We remark the relation between Conjecture 1 and other conjectures.

**Conjecture 2.** (Vandiver) Let \(p\) be an odd prime number. The class number of the field \(\mathbb{Q}(\zeta_p + \zeta_p^{-1})\) can not be divided by \(p\).

It is known that this conjecture holds for all prime numbers \(p < 12,000,000\).

**Conjecture 3.** (Quillen-Lichtenbaum) For any odd prime number \(p\), any integer \(i \geq 2\) and any number field \(L\), the \(p\)-adic Chern maps

\[
(1) \quad K_{2i-1}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_{et}(\text{Spec}(\mathcal{O}_L[1/p]), \mathbb{Z}_p(i))
\]

\[
(2) \quad K_{2i-2}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_{et}(\text{Spec}(\mathcal{O}_L[1/p]), \mathbb{Z}_p(i))
\]

are isomorphisms.

Soulé [17] \((2 \leq i \leq p)\) and Dwyer-Friedlander (for any \(i \geq 2\)) [3] proved that these maps are surjective. Further, Kurihara (under the assumption that there is only one prime of \(L(\mu_p)\) above \(p\)) [8] and Kahn [6] proved that the map \((2)\) is split surjective. We remark that this conjecture is considered to be true by the works of Voevodsky, Rost and others, but there is no published paper about it.

**Theorem 3.** (Kurihara[8], Mitchell[12]) Under Conjecture 3, Conjecture 1 is equivalent to Conjecture 2.

We conjecture on the \(K\)-groups of \(\mathbb{Z}[\mu_{p^{m+1}}]\) as follows. For any integer \(N \geq 0\), let \(\mu_N\) denote the group of all \(N\)-th roots of unity. For any abelian group \(G\), let \(\hat{G}\) denote the character group of \(G\). Put \(F = \mathbb{Q}(\mu_p)\). We denote by \(F^\infty\) the cyclotomic \(\mathbb{Z}_p\)-extension of \(F\), namely \(F^\infty = \bigcup_{m \geq 0} F_m\) with \(F_m = \mathbb{Q}(\mu_{p^{m+1}})\). Let \(\hat{k} : \text{Gal}(F^\infty/\mathbb{Q}) \to \hat{\mathbb{Z}}_p^\times\) be the cyclotomic character, namely \(\zeta^\tau = \zeta^{\hat{k}(\tau)}\) for any \(\tau \in \text{Gal}(F^\infty/\mathbb{Q})\) and any \(\zeta \in \bigcup_{m \geq 0} \mu_{p^{m+1}}\). Let \(\Delta = \text{Gal}(F/\mathbb{Q})\), \(\Gamma = \text{Gal}(F^\infty/F)\) and \(\Gamma_m = (F_m/F)\). By the isomorphism \(\text{Gal}(F^\infty/\mathbb{Q}) \simeq \Delta \times \Gamma\), the character \(\hat{k}\) is decomposed as follows.

\[
\hat{\text{Gal}}(F^\infty/\mathbb{Q}) \simeq \hat{\Delta} \times \hat{\Gamma}
\]

\[
\hat{k} = \omega \times \kappa
\]

with \(\omega : \Delta \to \mathbb{Z}_p^\times\) and \(\kappa : \Gamma \to \mathbb{Z}_p^\times\). The Galois group \(\Gamma\) is isomorphic to the multiplicative group \(1 + p\mathbb{Z}_p\), and the character \(\kappa\) gives an isomorphism \(\kappa : \Gamma \simeq 1 + p\mathbb{Z}_p\). We fix a topological generator \(\gamma\) of \(\Gamma\) such that \(\kappa(\gamma) = 1 + p\). By the correspondence \(\gamma \mapsto 1 + T\), we have an isomorphism \(\mathbb{Z}_p[[T]] \simeq \mathbb{Z}_p[[\gamma]]\). Put \(\Lambda = \mathbb{Z}_p[[\gamma]] \simeq \mathbb{Z}_p[[T]]\). For any character \(\chi\) of \(\Delta\), we denote by \(e_\chi\) the idempotent, namely

\[
e_\chi = \frac{1}{p-1} \sum_{\sigma \in \Delta} \chi^{-1}(\sigma)\sigma
\]

For any even character \(\chi(\neq 1)\) of \(\Delta\) and any character \(\psi\)
of $\Gamma_m$, let $f_\chi(T) \in \Lambda$ be the power series satisfying $L_p(s, \chi\psi) = f_\chi(\zeta_p\kappa(\gamma)^s - 1) = f_\chi(\zeta_p(1 + p)^s - 1)$, where $L_p(s, \chi\psi)$ is the Kubota-Leopoldt’s $p$-adic $L$-function and $\zeta_p = \psi(1 + p)^{-1} \in \mu_p$. For any $Z_p[[\text{Gal}(F_\infty/Q)]]$-module $V$ and any integer $n$, $V(n)$ means the Tate twist.

**Conjecture 4.** Put $m \geq 0$. For any odd prime $p$ and integers $i \geq 2$, $j$, we have

$$\left(K_{2i-2}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)^{\omega^j} \simeq \begin{cases} 0 & \text{if } i \equiv j \pmod{2} \text{ or } i \equiv j \pmod{p-1}, \\
\left(\Lambda/(f_{\omega^{-i-j}, \omega_{\omega^{-i-j}}^{(m)}})\right)_{(i-1)} & \text{if } i \equiv j \pmod{2} \text{ and } i \equiv j \pmod{p-1}, \\
\mu_{\mathbb{Z}_p}^{\omega^1} & \text{if } i \equiv j \pmod{2}, 
\end{cases}$$

where $g_{i}^{(m)} = (\gamma p^m - \kappa(\gamma p^m)) = (1 + T)^{p^m} - (1 + p)^{p^m} \in \Lambda(\simeq Z_p[[T]])$ and the finite group $\left(\Lambda/(f_{\omega^{-i-j}, \omega_{\omega^{-i-j}}^{(m)}})\right)_{(i-1)}$ has order $N_{p,i,j}$. For any integers $i, j$ with $i \equiv j \pmod{2}$, the power of $p : N_{p,i,j}$ and $D_{p,i,j}$ are given by $N_{p,i,j} \sim_p N_{p,i,j}$ and $D_{p,i,j} \sim_p D_{p,i,j}$ where

$$\prod_{\gamma \in \Gamma_m} L_p(1-i, \omega^{-i-j}) \sim_p \prod_{\gamma \in \Gamma_m} (B_{i, \omega^{-j}} \psi/i) = \frac{N_{p,i,j}}{D_{p,i,j}} \in \mathbb{Q}_p$$

with $N_{p,i,j}, D_{p,i,j} \in \mathbb{Z}_p$ and $v_p(N_{p,i,j}) = 0$ or $v_p(D_{p,i,j}) = 0$ ($N_{p,i,j}$ and $D_{p,i,j}$ are determined up to $p$-adic units). For any $\mathbb{Z}_p[\Delta]$-module $V$, we denote by $V^{\omega^j}$ its $\omega^j$-component, that is, $V^{\omega^j} = e_{\omega^j}V$. Further $' \sim_p'$ means that both sides have the same $p$-adic valuation.

Next we state certain conjecture on the values of $p$-adic $L$-function at integers.

**Conjecture 5.** $(C_{m,i})$ Let $m \geq 0$ and $i$ be integers. For any even character $\chi$ of $\Delta$ and any character $\psi$ of $\Gamma_m$, $L_p(1-i, \chi\psi)$ is not zero.

We remark $L_p(1-i, \psi) \neq 0$ for any $i$ and any $\psi \in \hat{\Gamma}_m$, and $(C_{m,i})$ for any $m \geq 0$ and $i \geq 0$ holds.

**Theorem 4.** Under Conjecture 3 and Conjecture 5 $(C_{m,1-i})$, Conjecture 4 for a pair $(m, i)$ is equivalent to Conjecture 2.

Before we give the proof of Theorem 4, we describe étale cohomology groups by using Iwasawa modules. For any integer $m \geq 0$, let $A_m$ be the $p$-primary component of the ideal class group of $F_m = \mathbb{Q}(\mu_{p^{m+1}})$. Put $X = \lim A_m$ (Iwasawa module).

From a result of Schneider ([16, 6.1] and [7, 3.1 and Remark 4] after Corollary 4.4]) and by using the fact that there is only one prime above $p$, we get the following lemma.
Lemma 1. For any integers $m \geq 0$ and $i$, we have the following natural isomorphisms.

1. $H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[1/p]), \mathbb{Z}_p(i)) \simeq X(i-1)_{\text{Gal}(F_\infty/\mathbb{Q})}$.

2. $H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[\mu_{p^m+1}, 1/p]), \mathbb{Z}_p(i)) \simeq X(i-1)_{\mathbb{F}_p}$.

Next, we introduce a quotient module of the Iwasawa module $X$.

Definition 1. For any integers $m \geq 0$, $i$ and $j$, let

$$X_{i,j}^{(m)} = \text{def} \quad X^{\omega_j}/(\gamma^m - \kappa_1(i) \gamma^m)X^{\omega_j}.$$

We can describe the étale cohomology groups by these quotient modules $X_{i,j}^{(m)}$.

The following lemma follows from Lemma 1 and the facts:

For any $\mathbb{Z}_p[[\text{Gal}(F_\infty/\mathbb{Q})]]$-module $V$ and any character $\chi$ of $\Delta$, we have

$$(V(i-1))^X = (V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i-1))^X = V^{\omega_j}(\gamma^m - \kappa_1(i) \gamma^m)X^{\omega_j}(i-1)$$

and

$$(V(i-1))_{\mathbb{F}_p} = (V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i-1))_{\mathbb{F}_p} = V^{\omega_j}(\gamma^m - \kappa_1(i) \gamma^m)X^{\omega_j}(i-1)$$

Proof of Theorem 4. First we can show the following lemma without Conjecture 2 (cf. [1]).

Lemma 2. For any integers $m \geq 0$, $i$ and $j$, we have the following natural isomorphisms.

1. $H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[1/p]), \mathbb{Z}_p(i)) \simeq X_{i,1-i}^{(0)}(i-1)$.

2. $H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[\mu_{p^m+1}, 1/p]), \mathbb{Z}_p(i)) \simeq X_{i,j+1}^{(m)}(i-1)$,

and hence

$$H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[\mu_{p^m+1}, 1/p]), \mathbb{Z}_p(i)) \simeq \bigoplus_{j=0}^{p-2} X_{i,j}^{(m)}(i-1).$$

Proof of Theorem 4. First we can show the following lemma without Conjecture 2 (cf. [1]).

Lemma 3. Under Conjecture 3 and Conjecture 5 $(C_{m,1-i})$, we have

$$(K_{2i-1}(\mathbb{Z}[\mu_{p^m+1}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{\omega_j} \simeq \begin{cases} \text{Hom}_{\mathbb{Z}_p}(\Lambda/(g_i^{(m)})_j e_\omega, Z_p(i)) & \text{if } i \not\equiv j \pmod{2}, \\ \mu_p^{\otimes i} & \text{if } i \equiv j \pmod{2}, \end{cases}$$

$$|(K_{2i-2}(\mathbb{Z}[\mu_{p^m+1}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{\omega_j}| = N_{p,i,j} \quad \text{if } i \equiv j \pmod{2}. $$
Next we show that Conjecture 4 for a pair \((m, i)\) is equivalent to Conjecture 2. We have the following equivalences.

Conjecture 2 \(\Leftrightarrow\) \(X^{\omega j} = 0\) for any even integer \(j\).

\[\Leftrightarrow (K_{2l-2}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega j} = 0.\]

for any integer \(j\) with \(i \equiv j \pmod{2}\).

We use Conjecture 3 and Lemma 2 (2) for the second equivalence. If the above equivalent facts hold, then we can show that \(X^{\omega j}\) for any odd integer \(j\) is generated by one element as \(\Lambda\)-module by the reflection theorem (cf. [18, §10.2]). Hence for any integer \(j\) with \(i \equiv j \pmod{2}\) and \(i \not\equiv j \pmod{p-1}\), we get the following surjection (note that if \(i \equiv j \pmod{p-1}\), then \(X^{(m)}_{i,j,i-1} = 0\)).

\[\Lambda/(f_{\omega^{-j}, g_{1,i-1}^{(m)}}) e_{\omega^{-j,i-1}} = X^{\omega^{-j+1}}_{i,j,i-1} / g_{1,i} X^{\omega^{-j+1}}_{i,j,i-1} = X^{(m)}_{i,j,i-1} (i-1),\]

we established the proof of Theorem 4. \(\Box\)

**Definition 2.** Let \(m \geq 0\), \(i\) and \(j\) be integers. For any \(\mathbb{Z}_p[[\text{Gal}(F_{\infty}/\mathbb{Q})]]\)-module \(V\), we define its \(k^i_m\)-component by

\[V^{k^i_m} = \{ v \in V \mid \gamma^{p^m} v = k^i(\gamma^{p^m}) v \}.\]

Next we give the orders of the modules \(X^{(m)}_{i,j}\). In the case \(j\) is odd, the order is given by using cyclotomic units. The following result is known as an instant consequence of a result of Mazur-Wiles (Iwasawa main conjecture) [10] and the structure theorem of local units modulo cyclotomic units due to Iwasawa [5].

Let \(E_m\) be the group of units of \(F_m\), \(E^{(1)}_m\) be the subgroup of \(E_m\) defined by \(E^{(1)}_m = \{ \varepsilon \in E_m \mid \varepsilon \equiv 1 \pmod{(1 - \zeta_{p^{m+1}})} \}\) and \(U_m\) be the group of local units of \(\mathbb{Q}_p(\mu_{p^{m+1}})\) which are congruent to 1 modulo the maximal ideal \((1 - \zeta_{p^{m+1}})\).

**Definition 3.** We define the group of cyclotomic units \(C_m\) by

\[C_m = \langle \{ \pm \zeta_{p^{m+1}}, 1 - \zeta_{p^{m+1}}^{a} \mid 1 \leq a \leq p^{m+1} - 1 \} \rangle \cap E_m\]

where \(\langle \{\ast\} \rangle\) denotes the multiplicative group generated by \(\{\ast\}\).

Let \(\Upsilon_m\) denotes the closure of the image of \(C_m \cap E^{(1)}_m\) in \(U_m\).

**Theorem 5.** (Mazur – Wiles [10], Iwasawa [5]) Let \(m_0 \geq 0\), \(i \geq 2\) be integers and \(j\) be an odd integer satisfying \(j \equiv 1 \pmod{p-1}\). There exists a constant \(C(m_0, i, j)\) which depends on \(m_0\), \(i\) and \(j\) such that for any integer \(m \geq C(m_0, i, j)\), we have

\[|X^{(m_0)}_{i,j}| = |(U_m/\Upsilon_m) \omega^{i-j} \zeta_{p^{m_0}}^i| \sim_p \prod_{\psi \in \Gamma_m} L_p(1 - i, \omega^{1-j} \psi) \sim_p \prod_{\psi \in \Gamma_m} (B_{i, \omega^{1-j} \psi/i}).\]
For any integers \( m \) and \( i \), we define the group of Gauss sums \( G_{m,i} \) by
\[
G_{m,i}^{(m_0)} = \langle \tau_\ell = \sum_{a=1}^{\ell-1} a \chi(a) \zeta_\ell^a \ | \ \lambda : \text{prime ideal of } F_m \text{ which divides a prime number } \ell \rangle
\]
where \( g_i^{(m_0)} = \gamma p^{m_0} \lambda^i (\gamma p^{m_0}) = (1 + T)^{p^{m_0}} - (1 + p)^{p^{m_0}} \in \Lambda(\mathbb{Z}_p[[T]]) \), and \( \chi : (\mathbb{Z}/(\mathbb{Z}))^\times \to \mu_{p^{m+1}} \) is given by \( \chi(a) \equiv a^{-1}/p^{m+1} \) (mod \( \lambda \)), and \( \langle \{ * \} \rangle \mathbb{Z}_p \)
means the \( \mathbb{Z}_p \)-module generated by \( \{ * \} \).

Let \( \overline{G}_{m,i}^{(m_0)} \) denote the image of \( G_{m,i}^{(m_0)} \cap (F_m^{\times} \otimes \mathbb{Z}_p) \) in \( \mathbb{Q}_p(\mu_{p^{m+1}}) \) (Note that \( \overline{G}_{m,i}^{(m_0)} \subseteq \mathcal{U}_m \)).

We get the following result which is an analogue of Theorem 5.

**Theorem 6.** Let \( m_0 \geq 0, i \geq 2 \) be integers and \( j \) be an even integer. There exists a constant \( C(m_0,i,j) \) which depends on \( m_0, i \) and \( j \) such that for any integer \( m \geq C(m_0,i,j) \), we have
\[
|X_{i,j}^{(m_0)}| = |(\mathcal{U}_m/\overline{G}_{m,i}^{(m_0)} + I_m \mathcal{U}_m) \omega^{-j} || \mathcal{A}_{\omega^{-j} \kappa^{(m_0)}} \omega^{-j} | A_{\omega^{-j} \kappa^{(m_0)}} | / |A_{\omega^{-j}} |.
\]

**Proof.** Let \( L_m/F_m \) be the maximal unramified abelian pro-\( p \)-extension and \( M_m/F_m \) be the maximal abelian pro-\( p \)-extension unramified outside \( p \). Put \( Y_m = \text{Gal}(M_m/F_m) \) and \( Z_m = \text{Gal}(M_m/L_m) \). We consider the following commutative diagram.

\[
\begin{array}{ccccccccc}
Y_m^{\omega^{-j} \kappa^{(m_0)}} & \rightarrow & 0 \\
\downarrow & & \\
Z_m^{\omega^{-j} \kappa^{(m_0)}} & \rightarrow & Y_m^{\omega^{-j}} & \rightarrow & A_m^{\omega^{-j} \kappa^{(m_0)}} & \rightarrow & 0 \\
\downarrow g_i^{(m_0)} & & \downarrow g_i^{(m_0)} & & \downarrow g_i^{(m_0)} & & \downarrow \\
0 & \rightarrow & Z_m^{\omega^{-j}} & \rightarrow & Y_m^{\omega^{-j}} & \rightarrow & A_m^{\omega^{-j}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z_m^{\omega^{-j}} & \rightarrow & Y_m^{\omega^{-j}} & \rightarrow & A_m^{\omega^{-j}} & \rightarrow & 0 \\
\end{array}
\]

with \( g_i^{(m_0)} = \gamma p^{m_0} - \kappa^i (\gamma p^{m_0}) \).
We can show the following facts with respect to the above diagram. For the precise proof, see [1].

1. The left vertical map $Z_{\omega}^{\omega - j} \to Z_{\omega}^{\omega - j}$ given by multiplying by $g_{m}^{(m_0)}$ is injective. This fact follows from the isomorphism $Z_{\omega}^{\omega - j} \simeq \Lambda/(\omega_m)$ ($\omega_m = \gamma^p - 1$) and the fact that the polynomials $\omega_m$ and $g_{m}^{(m_0)}$ are relatively prime.

2. The image of the bottom horizontal map is given by
   
   $$\text{Im}\{A_{\omega}^{\omega - j}/g_{m}^{(m_0)}Z_{\omega}^{\omega - j}\} \simeq (G_{m,i} + I_m U_m)/I_m U_m.$$  

3. From a result of Nguyen Quang Do [13, Theorem 1.1] with respect to the $\mathbb{Z}_p$-torsion $F_{\text{tor}} \mathbb{Z}_p(\mathbb{Z}/p(1)) \simeq \Gamma_m^{\omega - j}$, we get an isomorphism $\mathbb{Z}_p(X_{1,j}^{(m_0)}, \mathbb{Q}_p/\mathbb{Z}_p(1))$ for any enough large integer $m$.

From these facts and the snake lemma, we get the following exact sequence.

$$0 \to \text{Hom}_\mathbb{Z}(X_{1,j}^{(m_0)}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to A_{\omega}^{\omega - j}/g_{m}^{(m_0)}Z_{\omega}^{\omega - j} \to (U_m/I_m U_m)^{\omega - j} \to 0$$

By Iwasawa main conjecture, we have $|(U_m/I_m U_m)^{\omega - j}| = |A_{\omega}^{\omega - j}|$, and hence we get the conclusion. □

Putting Lemma 2, Theorem 5 and Theorem 6 together, we get the following explicit formulas of the orders of étale cohomology groups $H^2_{et}(\text{Spec}(\mathbb{O}_L[1/p]), \mathbb{Z}_p(i))$ for $L = \mathbb{Q} \text{ or } \mathbb{Q}(\mu_{p+1})$. Note that these formulas give also the orders of the $K$-groups for even degree under Conjecture 3.

**Corollary 1.** (1) Let $m_0 \geq 0$, $i \geq 2$ be integers satisfying $i \equiv 0 \pmod{p - 1}$. There exists a constant $C(i)$ such that for any integer $m \geq C(i)$, we have

$$|H^2_{et}(\text{Spec}(\mathbb{Z}[1/p]), \mathbb{Z}_p(i))| = \begin{cases} |(U_m/C_m)^{\omega - j}/g_{m}^{(m_0)}| & \text{if } i \text{ is even,} \\ |(U_m/C_m)^{(0)} + I_m U_m)^{\omega - j}| & |A_m^{\omega - j}/|A_m^{\omega - j}|| & \text{if } i \text{ is odd.} \end{cases}$$
(2) Let \( m_0 \geq 0 \), \( i \geq 2 \) and \( j \) be integers satisfying \( i - j \equiv 0 (\text{mod } p - 1) \). There exists a constant \( C(m_0, i, j) \) such that for any integer \( m \geq C(m_0, i, j) \), we have

\[
|H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[\mu_{p^{m_0+1}}, 1/p]), Z_p(i))| = \begin{cases} 
|\mathcal{U}_m/\mathcal{G}_{m, i}| & \text{if } i \equiv j \pmod{2}, \\
|\mathcal{U}_m/\mathcal{G}_{m, i}^{(m_0)} + I_m \mathcal{U}_m| \cdot \left| \frac{A_m^{\omega_{i-j}}}{A_m} \right| & \text{if } i \not\equiv j \pmod{2}.
\end{cases}
\]

If we assume the non-vanishingness of the \( p \)-adic \( L \)-function at an integer, we get the following corollary.

**Corollary 2.** (1) Let \( m_0 \geq 0 \), \( i \geq 2 \) be integers satisfying \( i \equiv 0, 1 \pmod{p - 1} \). Assume \( L_p(i, \omega^{1-i}) \neq 0 \). There exists a constant \( C(i) \) such that for any integer \( m \geq C(i) \), we have

\[
|H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[1/p], Z_p(i)))| \sim_p \begin{cases} 
L_p(1 - i, \omega^i) \sim_p B_{i/i} & \text{if } i \text{ is even}, \\
|\mathcal{U}_m/\mathcal{G}_{m, i}^{(0)} + I_m \mathcal{U}_m| \cdot \left| \frac{L_p(i, \omega^{1-i})}{\prod_{\psi \in \hat{\Gamma}_m} B_{1, \omega^{-1}} \psi} \right| & \text{if } i \text{ is odd}.
\end{cases}
\]

(2) Let \( m_0 \geq 0 \), \( i \geq 2 \), \( j \) be integers satisfying \( i - j \equiv 0, 1 \pmod{p - 1} \). Assume \( L_p(i, \omega^{j-i+1}) \neq 0 \) for all \( \psi \in \hat{\Gamma}_m \). There exists an constant \( C(m_0, i, j) \) such that for any integer \( m \geq C(m_0, i, j) \), we have

\[
|H^2_{\text{ét}}(\text{Spec}(\mathbb{Z}[\mu_{p^{m_0+1}}, 1/p]), Z_p(i))| \sim_p \begin{cases} 
\prod_{\psi \in \hat{\Gamma}_m} L_p(1 - i, \omega^{j-i}) \sim_p \prod_{\psi \in \hat{\Gamma}_m} (B_{i, \omega^{-1}} \psi/i) & \text{if } i \equiv j \pmod{2}, \\
|\mathcal{U}_m/\mathcal{G}_{m, i}^{(m_0)} + I_m \mathcal{U}_m| \cdot \left| \frac{\prod_{\psi \in \hat{\Gamma}_m} L_p(i, \omega^{j-i+1})}{\prod_{\psi \in \hat{\Gamma}_m} B_{1, \omega^{-1}} \psi} \right| & \text{if } i \not\equiv j \pmod{2}.
\end{cases}
\]

At the end of this article, we state the relation between Corollary 1, 2 and Conjecture 4. Under Conjecture 5 (\( C_{m,1-i} \)) and Conjecture 3, we get the following equivalences.
Conjecture 4 for a pair \((m, i)\)

\[
\iff \left| \frac{\omega^{i-j}_{m, i}}{A^{\omega_{i-j}}_{m, i}} \right| \sim_p 1
\]

for any integer \(j\) such that \(i \equiv j \pmod{2}\).

\[
\iff \left| \frac{\prod_{\psi \in \hat{\Gamma}_m} L_p(i, \omega^{1-i+j}_{\psi})}{\prod_{\psi \in \hat{\Gamma}_m} B_{1, \omega^{i-j}_{\psi}}} \right| \sim_p 1
\]

for any integer \(j\) such that \(i \equiv j \pmod{2}\).

References


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