Trends in Mathematics
Information Center for Mathematical Sciences
Volume 9, Number 1, June, 2006, Pages 39–48

ON THE K-GROUPS OF ALGEBRAIC INTEGERS OF CYCLOTOMIC FIELDS

MIHO AOKI

Let L be a number field, $\mathcal{O}_L$ be the ring of algebraic integers of L. For any integer $n \geq 0$, we denote the n-th Quillen K-group of $\mathcal{O}_L$ by $K_n(\mathcal{O}_L)$, especially we have $K_0(\mathcal{O}_L) \simeq \mathbb{Z} \oplus Pic(\mathcal{O}_L)$ and $K_1(\mathcal{O}_L) \simeq \mathcal{O}_L^{\times}$. The following facts on the K-groups are well known.

Theorem 1. (Quillen[14]) *For any integer $n \geq 0$, $K_n(\mathcal{O}_L)$ is a finitely generated abelian group.*

Theorem 2. (Borel [2]) *For any integer $n \geq 0$, we have*

$$\mathrm{rank}_{\mathbb{Z}} K_n(\mathcal{O}_L) = \begin{cases} 0 & \text{if } n \geq 2 \text{ and even}, \\ r_1 + r_2 & \text{if } n \geq 5 \text{ and } n \equiv 1 \pmod 4, \\ r_2 & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where r_1 or r_2 is the number of the real or complex places of L.

Example 1. *The structures of the K-groups of $\mathbb{Z}$ for lower degree are known as follows. $K_0(\mathbb{Z}) \simeq \mathbb{Z}$, $K_1(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, $K_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$* (Milnor [11])*, $K_3(\mathbb{Z}) \simeq \mathbb{Z}/48\mathbb{Z}$* (Lee-Szczarba [9])*, $K_4(\mathbb{Z}) = 0$* (Rognes [15])*, $K_5(\mathbb{Z}) \simeq \mathbb{Z}$* (Elbaz Vincent-Gangl-Soulé [4])*.*

Kurihara [8] and Mitchell [12] conjectured independently the following statement.

Conjecture 1. (Kurihara [8,Conjecture 3.2], Mitchell [12, 6.15]) *For any integer $n \geq 0$, we have*

$$K_{4n}(\mathbb{Z}) \simeq' 0 \qquad (n \geq 1)$$

$$K_{4n+1}(\mathbb{Z}) \simeq' \mathbb{Z} \qquad (n \geq 1)$$

$$K_{4n+2}(\mathbb{Z}) \simeq' \mathbb{Z}/N_{2n+2}\mathbb{Z}$$

$$K_{4n+3}(\mathbb{Z}) \simeq' \mathbb{Z}/D_{2n+2}\mathbb{Z}$$

where for an even integer k, the positive integers N_k and D_k are given by $\zeta(1-k) = -\dfrac{B_k}{k}$ $= (-1)^{k/2}\dfrac{N_k}{D_k}$ *with* $(N_k, D_k) = 1$. *The Bernoulli number B_k is the same notation as in* [8] *or* [18], *and '$\simeq$' means an isomorphism up to 2-torsion groups.*

Note that Mitchell [12] give the statement including 2-torsion groups. We remark the relation between Conjecture 1 and other conjectures.

Conjecture 2. (Vandiver) *Let p be an odd prime number. The class number of the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ can not be divided by p.*

It is known that this conjecture holds for all prime numbers $p < 12,000,000$.

Conjecture 3. (Quillen-Lictenbaum) *For any odd prime number p, any integer $i \geq 2$ and any number field L, the p-adic Chern maps*

$$\begin{aligned} (1)\ K_{2i-1}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p &\rightarrow H^1_{\acute{e}t}(Spec(\mathcal{O}_L[1/p]), \mathbb{Z}_p(i)) \\ (2)\ K_{2i-2}(\mathcal{O}_L) \otimes_{\mathbb{Z}} \mathbb{Z}_p &\rightarrow H^2_{\acute{e}t}(Spec(\mathcal{O}_L[1/p]), \mathbb{Z}_p(i)) \end{aligned}$$

are isomorphisms.

Soulé [17] ($2 \leq i \leq p$) and Dwyer-Friedlander (for any $i \geq 2$) [3] proved that these maps are surjective. Further, Kurihara (under the assumption that there is only one prime of $L(\mu_p)$ above p) [8] and Kahn [6] proved that the map (2) is split surjective. We remark that this conjecture is considered to be true by the works of Voevodsky, Rost and others, but there is no piblished paper about it.

Theorem 3. (Kurihara[8], Mitchell[12]) *Under Conjecture 3, Conjecture 1 is equivalent to Conjecture 2.*

We conjecture on the K-groups of $\mathbb{Z}[\mu_{p^{m+1}}]$ as follows. For any integer $N \geq 0$, let μ_N denote the group of all N-th roots of unity. For any abelian group G, let $\widehat{G}$ denote the character group of G. Put $F = \mathbb{Q}(\mu_p)$. We denote by F_∞ the cyclotomic $\mathbb{Z}_p$-extension of F, namely $F_\infty = \cup_{m\geq 0} F_m$ with $F_m = \mathbb{Q}(\mu_{p^{m+1}})$. Let $\tilde{\kappa}$:Gal$(F_\infty/\mathbb{Q}) \rightarrow \mathbb{Z}_p^\times$ be the cyclotomic character, namely $\zeta^\tau = \zeta^{\tilde{\kappa}(\tau)}$ for any $\tau \in$Gal$(F_\infty/\mathbb{Q})$ and any $\zeta \in \cup_{m\geq 0}\mu_{p^{m+1}}$. Let $\Delta =$Gal$(F/\mathbb{Q})$, $\Gamma =$Gal(F_∞/F) and $\Gamma_m = (F_m/F)$. By the isomorphism Gal$(F_\infty/\mathbb{Q}) \simeq \Delta \times \Gamma$, the character $\tilde{\kappa}$ is decomposed as follows.

$$\begin{array}{ccc} \widehat{\mathrm{Gal}}(F_\infty/\mathbb{Q}) & \simeq & \widehat{\Delta} \times \widehat{\Gamma} \\ \\ \tilde{\kappa} & = & \omega \times \kappa \end{array}$$

with $\omega : \Delta \rightarrow \mathbb{Z}_p^\times$ and $\kappa : \Gamma \rightarrow \mathbb{Z}_p^\times$. The Galois group Γ is isomorophic to the multiplicative group $1 + p\mathbb{Z}_p$, and the character κ gives an isomorphism $\kappa : \Gamma \simeq 1 + p\mathbb{Z}_p$. We fix a topological generator γ of Γ such that $\kappa(\gamma) = 1 + p$. By the correspondence $\gamma \mapsto 1 + T$, we have an isomorphism $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$. Put $\Lambda = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$. For any character χ of Δ, we denote by e_χ the idempotent, namely $e_\chi = \dfrac{1}{p-1}\sum_{\sigma\in\Delta}\chi^{-1}(\sigma)\sigma$. For any even character $\chi(\neq \mathbf{1})$ of Δ and any character ψ

of Γ_m, let $f_\chi(T) \in \Lambda$ be the power series satisfying $L_p(s, \chi\psi) = f_\chi(\zeta_\psi \kappa(\gamma)^s - 1) = f_\chi(\zeta_\psi(1+p)^s - 1)$, where $L_p(s, \chi\psi)$ is the Kubota-Leopoldt's p-adic L-function and $\zeta_\psi = \psi(1+p)^{-1} \in \mu_{p^m}$. For any $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/\mathbb{Q})]]$-module V and any integer n, $V(n)$ means the Tate twist.

Conjecture 4. *Put $m \geq 0$. For any odd prime p and integers $i \geq 2$, j, we have*

$$(K_{2i-2}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j} \simeq \begin{cases} 0 & \text{if } i \not\equiv j \pmod 2 \text{ or } i \equiv j \pmod{p-1}, \\ \left(\Lambda/(f_{\omega^{i-j}}, g_{1-i}^{(m)})e_{\omega^{j-i+1}}\right)(i-1) & \\ \qquad \text{if } i \equiv j \pmod 2 \text{ and } i \not\equiv j \pmod{p-1}, \end{cases}$$

$$(K_{2i-1}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j} \simeq \begin{cases} \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/(g_i^{(m)})e_{\omega^{i-j}}, \mathbb{Z}_p(i)) & \text{if } i \not\equiv j \pmod 2, \\ \mu_{D_{p,i,j}}^{\otimes i} & \text{if } i \equiv j \pmod 2, \end{cases}$$

where $g_i^{(m)} = (\gamma^{p^m} - \kappa^i(\gamma^{p^m}))$ $(= (1+T)^{p^m} - (1+p)^{ip^m}) \in \Lambda(\simeq \mathbb{Z}_p[[T]])$ and the finite group $\left(\Lambda/(f_{\omega^{i-j}}, g_{1-i}^{(m)})e_{\omega^{j-i+1}}\right)(i-1)$ has order $N_{p,i,j}$. For any integers i, j with $i \equiv j \pmod 2$, the power of p : $N_{p,i,j}$ and $D_{p,i,j}$ are given by $N_{p,i,j} \sim_p \mathcal{N}_{p,i,j}$ and $D_{p,i,j} \sim_p \mathcal{D}_{p,i,j}$ where

$$\prod_{\psi \in \widehat{\Gamma}_m} L_p(1-i, \omega^{i-j}\psi) \sim_p \prod_{\psi \in \widehat{\Gamma}_m} (B_{i,\omega^{-j}\psi}/i) = \frac{\mathcal{N}_{p,i,j}}{\mathcal{D}_{p,i,j}} \quad \in \mathbb{Q}_p$$

with $\mathcal{N}_{p,i,j}$, $\mathcal{D}_{p,i,j} \in \mathbb{Z}_p$ and $v_p(\mathcal{N}_{p,i,j}) = 0$ or $v_p(\mathcal{D}_{p,i,j}) = 0$ ($\mathcal{N}_{p,i,j}$ and $\mathcal{D}_{p,i,j}$ are determined up to p-adic units). For any $\mathbb{Z}_p[\Delta]$-module V, we denote by V^{ω^j} its ω^j-component, that is, $V^{\omega^j} = e_{\omega^j}V$. Further $' \sim_p'$ means that both sides have the same p-adic valuation.

Next we state certain conjecture on the values of p-adic L-function at integers.

Conjecture 5. $(\mathrm{C}_{m,i})$ *Let $m \geq 0$ and i be integers. For any even character χ of Δ and any character ψ of Γ_m, $L_p(1-i, \chi\psi)$ is not zero.*

We remark $L_p(1-i, \psi) \neq 0$ for any i and any $\psi \in \widehat{\Gamma}_m$, and $(\mathrm{C}_{m,i})$ for any $m \geq 0$ and $i \geq 0$ holds.

Theorem 4. *Under Conjecture 3 and Conjecture 5 $(\mathrm{C}_{m,1-i})$, Conjecture 4 for a pair (m, i) is equivalent to Conjecture 2.*

Before we give the proof of Theorem 4, we describe étale cohomology groups by using Iwasawa modules. For any integer $m \geq 0$, let A_m be the p-primary component of the ideal class group of $F_m = \mathbb{Q}(\mu_{p^{m+1}})$. Put $X = \varprojlim A_m$ (Iwasawa module).

From a result of Schneider ([16, 6.1] and [7, 3.1 and Remark (4) after Corollary 4.4]) and by using the fact that there is only one prime above p, we get the following lemma.

Lemma 1. *For any integers $m \geq 0$ and i, we have the following natural isomorphisms.*

(1) $H^2_{ét}(\mathrm{Spec}(\mathbb{Z}[1/p]), \mathbb{Z}_p(i)) \simeq X(i-1)_{\mathrm{Gal}(F_\infty/\mathbb{Q})}$.

(2) $H^2_{ét}(\mathrm{Spec}(\mathbb{Z}[\mu_{p^{m+1}}, 1/p]), \mathbb{Z}_p(i)) \simeq X(i-1)_{\Gamma^{p^m}}$.

Next, we introduce a quotient module of the Iwasawa module X.

Definition 1. *For any integers $m \geq 0$, i and j, let*

$$X^{(m)}_{i,j} =_{\mathrm{def}} \quad X^{\omega^j}/(\gamma^{p^m} - \kappa^{1-i}(\gamma^{p^m}))X^{\omega^j}.$$

We can describe the étale cohomology groups by these quotient modules $X^{(m)}_{i,j}$. The following lemma follows from Lemma 1 and the facts:

For any $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/\mathbb{Q})]]$-module V and any character χ of Δ, we have

$$(V(i-1))^\chi = (V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i-1))^\chi = V^{\chi\omega^{1-i}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i-1) = V^{\chi\omega^{1-i}}(i-1)$$

and

$$(V(i-1))_{\Gamma^{p^m}} = (V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i-1))_{\Gamma^{p^m}} = V/(\gamma^{p^m} - \kappa^{1-i}(\gamma^{p^m}))V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i-1)$$

$$= \left(V/(\gamma^{p^m} - \kappa^{1-i}(\gamma^{p^m}))V\right)(i-1).$$

Lemma 2. *For any integers $m \geq 0$, i and j, we have the following natural isomorphisms.*

(1) $H^2_{ét}(\mathrm{Spec}(\mathbb{Z}[1/p]), \mathbb{Z}_p(i)) \simeq X^{(0)}_{i,1-i}(i-1)$.

(2) $H^2_{ét}(\mathrm{Spec}(\mathbb{Z}[\mu_{p^{m+1}}, 1/p]), \mathbb{Z}_p(i))^{\omega^j} \simeq X^{(m)}_{i,j-i+1}(i-1)$,

and hence

$$H^2_{ét}(\mathrm{Spec}(\mathbb{Z}[\mu_{p^{m+1}}, 1/p]), \mathbb{Z}_p(i)) \simeq \bigoplus_{j=0}^{p-2} X^{(m)}_{i,j}(i-1).$$

Proof of Theorem 4. First we can show the folloing lemma without Conjecture 2 (cf. [1]).

Lemma 3. *Under Conjecture 3 and Conjecture 5* ($\mathrm{C}_{m,1-i}$)*, we have*

$$(K_{2i-1}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j} \simeq \begin{cases} \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/(g_i^{(m)})e_{\omega^{i-j}}, \mathbb{Z}_p(i)) & \textit{if } i \not\equiv j \pmod 2, \\ \mu^{\otimes i}_{D_{p,i,j}} & \textit{if } i \equiv j \pmod 2, \end{cases}$$

$$|(K_{2i-2}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j}| = N_{p,i,j} \qquad \textit{if } i \equiv j \pmod 2.$$

Next we show that Conjecture 4 for a pair (m, i) is equivalent to Conjecture 2. We have the following equivalences.

$$\begin{aligned}\text{Conjecture 2} \Leftrightarrow \quad & X^{\omega^j} = 0 \text{ for any even integer } j. \\ \Leftrightarrow \quad & (K_{2i-2}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j} = 0. \\ & \text{for any integer } j \text{ with } i \not\equiv j \pmod 2.\end{aligned}$$

We use Conjecture 3 and Lemma 2 (2) for the second equivalence. If the above equivalent facts hold, then we can show that X^{ω^j} for any odd integer j is generated by one element as Λ-module by the reflection theorem (cf. [18, §10.2]). Hence for any integer j with $i \equiv j \pmod 2$ and $i \not\equiv j \pmod{p-1}$, we get the following surjection (note that if $i \equiv j \pmod{p-1}$, then $X^{(m)}_{i,j-i+1} = 0$).

$$\Lambda/(f_{\omega^{i-j}}, g^{(m)}_{1-i})e_{\omega^{j-i+1}} \twoheadrightarrow X^{\omega^{j-i+1}}/g^{(m)}_{1-i}X^{\omega^{j-i+1}} = X^{(m)}_{i,j-i+1}$$

We can easily show that both the modules have tha same order $N_{p,i,j}$. Hence we get the isomorphism $(K_{2i-2}(\mathbb{Z}[\mu_{p^{m+1}}])\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\omega^j} \simeq X^{(m)}_{i,j-i+1}(i-1) \simeq \Big(\Lambda/(f_{\omega^{i-j}}, g^{(m)}_{1-i})e_{\omega^{j-i+1}}\Big)$ $(i-1)$, and we established the proof of Theorem 4. $\square$

Definition 2. *Let $m \geq 0$, i and j be integers. For any $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/\mathbb{Q})]]$-module V, we define its κ^i_m-component by*

$$V^{\kappa^i_m} = \{\ v \in V \mid \gamma^{p^m} v = \kappa^i(\gamma^{p^m})v\ \}.$$

Next we give the orders of the modules $X^{(m)}_{i,j}$. In the case j is odd, the order is given by using cyclotomic units. The following result is known as an instant consequence of a result of Mazur-Wiles (Iwasawa main conjecture) [10] and the structure theorem of local units modulo cyclotomic units due to Iwasawa [5].

Let E_m be the group of units of F_m, $E^{(1)}_m$ be the subgroup of E_m defined by $E^{(1)}_m = \{\varepsilon \in E_m \mid \varepsilon \equiv 1 \pmod{(1-\zeta_{p^{m+1}})}\}$ and $\mathcal{U}_m$ be the group of local units of $\mathbb{Q}_p(\mu_{p^{m+1}})$ which are congruent to 1 modulo the maximal ideal $(1-\zeta_{p^{m+1}})$.

Definition 3. *We define the group of cyclotomic units $\mathcal{C}_m$ by*

$$\mathcal{C}_m = \langle\{\ \pm\zeta_{p^{m+1}}, 1-\zeta^a_{p^{m+1}} \mid 1 \leq a \leq p^{m+1}-1\ \}\rangle \cap E_m$$

where $\langle\{*\}\rangle$ denotes the multiplicative group generated by $\{*\}$.

Let $\overline{\mathcal{C}}_m$ denotes the closure of the image of $\mathcal{C}_m \cap E^{(1)}_m$ in $\mathcal{U}_m$.

Theorem 5. (Mazur – Wiles [10], Iwasawa [5]) *Let $m_0 \geq 0$, $i \geq 2$ be integers and j be an odd integer satisfying $j \not\equiv 1 \pmod{p-1}$. There exists a constant $C(m_0, i, j)$ which depends on m_0, i and j such that for any integer $m \geq C(m_0, i, j)$, we have*

$$|X^{(m_0)}_{i,j}| = |(\mathcal{U}_m/\overline{\mathcal{C}}_m)^{\omega^{1-j}\kappa^i_{m_0}}| \sim_p \prod_{\psi\in\widehat{\Gamma}_{m_0}} L_p(1-i, \omega^{1-j}\psi) \sim_p \prod_{\psi\in\widehat{\Gamma}_{m_0}} (B_{i,\omega^{1-j-i}\psi}/i).$$

Next, we consider the case j is even. In this case, the order of $X_{i,j}^{(m)}$ is given by using Gauss sums. Let

$$\theta_m = \frac{1}{p^{m+1}} \sum_{\substack{a=0 \\ (a,p)=1}}^{p^{m+1}} a\sigma_a^{-1} \in \mathbb{Q}_p[\mathrm{Gal}(F_m/\mathbb{Q})]$$

and

$$I_m = \mathbb{Z}_p[\mathrm{Gal}(F_m/\mathbb{Q})] \cap \theta_m \mathbb{Z}_p[\mathrm{Gal}(F_m/\mathbb{Q})]$$

be Stickelberger element and Stickelberger ideal respectively.

Definition 4. *For any integers m_0, m $(m_0 \leq m)$ and i, we define the group of Gauss sums $\mathcal{G}_{m,i}^{(m_0)}$ by*

$$\mathcal{G}_{m,i}^{(m_0)} = \langle \{ \ \tau_\lambda = \sum_{a=1}^{\ell-1} \chi_\lambda(a)\zeta_\ell^a \mid \lambda : \textit{prime ideal of } F_m \textit{ which divides a prime number } \ell$$
$$\textit{satisfying } \ell \equiv 1 \pmod{p^{m+1}} \text{ and } \tau^{g_i^{(m_0)}} \in I_m^-(F_m(\mu_\ell)^\times \otimes \mathbb{Z}_p) \ \}\rangle_{\mathbb{Z}_p},$$

where $g_i^{(m_0)} = \gamma^{p^{m_0}} - \kappa^i(\gamma^{p^{m_0}})$ $(= (1+T)^{p^{m_0}} - (1+p)^{ip^{m_0}}) \in \Lambda(\simeq \mathbb{Z}_p[[T]])$, and $\chi_\lambda : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mu_{p^{m+1}}$ is given by $\chi_\lambda(a) \equiv a^{-(\ell-1)/p^{m+1}} \pmod{\lambda}$, and $\langle\{*\}\rangle_{\mathbb{Z}_p}$ means the $\mathbb{Z}_p$-module generated by $\{*\}$.

Let $\overline{\mathcal{G}}_{m,i}^{(m_0)}$ denote the image of $\mathcal{G}_{m,i}^{(m_0)} \cap (F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ in $\mathbb{Q}_p(\mu_{p^{m+1}})$ (Note that $\overline{\mathcal{G}}_{m,i}^{(m_0)} \subset \mathcal{U}_m$).

We get the following result which is an analogue of Theorem 5.

Theorem 6. *Let $m_0 \geq 0$, $i \geq 2$ be integers and j be an even integer. There exists a constant $C(m_0,i,j)$ which depends on m_0, i and j such that for any integer $m \geq C(m_0,i,j)$, we have*

$$|X_{i,j}^{(m_0)}| = |(\mathcal{U}_m/\overline{\mathcal{G}}_{m,i}^{(m_0)} + I_m\mathcal{U}_m)^{\omega^{1-j}}||A_m^{\omega^{1-j}\kappa_{m_0}^i}|/|A_m^{\omega^{1-j}}|.$$

Proof. Let L_m/F_m be the maximal unramified abelian pro-p-extension and M_m/F_m be the maximal abelian pro-p-extension unramified outside p. Put $Y_m = \mathrm{Gal}(M_m/F_m)$ and $Z_m = \mathrm{Gal}(M_m/L_m)$. We consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & Y_m^{\omega^{1-j}\kappa_{m_0}^i} & \to & A_m^{\omega^{1-j}\kappa_{m_0}^i} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \to & & Z_m^{\omega^{1-j}} & \to & Y_m^{\omega^{1-j}} & \to & A_m^{\omega^{1-j}} & \to 0 \\
 & & \downarrow g_i^{(m_0)} & & \downarrow g_i^{(m_0)} & & \downarrow g_i^{(m_0)} & & \\
0 \to & & Z_m^{\omega^{1-j}} & \to & Y_m^{\omega^{1-j}} & \to & A_m^{\omega^{1-j}} & \to 0 \\
 & & \downarrow & & & & & & \\
 & \to & Z_m^{\omega^{1-j}}/g_i^{(m_0)}Z_m^{\omega^{1-j}} & & & & & &
\end{array}$$

with $g_i^{(m_0)} = \gamma^{p^{m_0}} - \kappa^i(\gamma^{p^{m_0}})$.

We can show the following facts with respect to the above diagram. For the precise proof, see [1].

① The left vertical map $Z_m^{\omega^{1-j}} \to Z_m^{\omega^{1-j}}$ given by multiplying by $g_i^{(m_0)}$ is injective. This fact follows from the isomorphism $Z_m^{\omega^{1-j}} \simeq \Lambda/(\omega_m)$ $(\omega_m = \gamma^{p^m} - 1)$ and the fact that the polynomials ω_m and $g_i^{(m_0)}$ are relatively prime.

② The image of the bottom horizontal map is given by

$$\mathrm{Im}\{A_m^{\omega^{1-j}\kappa_{m_0}^i} \to Z_m^{\omega^{1-j}}/g_i^{(m_0)} Z_m^{\omega^{1-j}}\} \simeq (\overline{\mathcal{G}}_{m,i}^{(m_0)} + I_m\mathcal{U}_m/I_m\mathcal{U}_m)^{\omega^{1-j}}.$$

③ From a result of Nguyen Quang Do [13, Theorem 1.1] with respect to the $\mathbb{Z}_p$-torsionF $\mathrm{tor}_{\mathbb{Z}_p}(Y_m) \simeq \mathrm{Hom}_{\mathbb{Z}_p}(X, \mathbb{Q}_p/\mathbb{Z}_p(1))^{\Gamma^{p^m}}$, we get an isomorphism $Y_m^{\omega^{1-j}\kappa_{m_0}^i} \simeq \mathrm{Hom}_{\mathbb{Z}_p}(X_{i,j}^{(m_0)}, \mathbb{Q}_p/\mathbb{Z}_p(1))$ for any enough large integer m.

From these facts and the snake lemma, we get the following exact sequence.

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathbb{Z}_p}(X_{i,j}^{(m_0)}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \to A_m^{\omega^{1-j}\kappa_{m_0}^i} &\to (\mathcal{U}_m/I_m\mathcal{U}_m)^{\omega^{1-j}} \\ &\to (\mathcal{U}_m/\overline{\mathcal{G}}_{m,i}^{(m_0)} + I_m\mathcal{U}_m)^{\omega^{1-j}} \to 0 \end{aligned}$$

By Iwasawa main conjecture, we have $|(\mathcal{U}_m/I_m\mathcal{U}_m)^{\omega^{1-j}}| = |A_m^{\omega^{1-j}}|$, and hence we get the conclusion. □

Putting Lemma 2, Theorem 5 and Theorem 6 together, we get the following explicit formulas of the orders of étale cohomology groups $H^2_{ét}(\mathrm{Spec}(\mathcal{O}_L[1/p]), \mathbb{Z}_p(i))$ for $L = \mathbb{Q}$ or $\mathbb{Q}(\mu_{p^{m+1}})$. Note that these formulas give also the orders of the K-groups for even degree under Conjecture 3.

Corollary 1. (1) *Let $m_0 \geq 0$, $i \geq 2$ be integers satisfing $i \not\equiv 0 \pmod{p-1}$. There exists a constant $C(i)$ such that for any integer $m \geq C(i)$, we have*

$$|H^2_{ét}(\mathrm{Spec}(\mathbb{Z}[1/p]), \mathbb{Z}_p(i))| = \begin{cases} |(\mathcal{U}_m/\overline{\mathcal{C}}_m)^{\omega^i\kappa_0^i}| & \text{if } i \text{ is even,} \\ \\ |(\mathcal{U}_m/\overline{\mathcal{G}}_{m,i}^{(0)} + I_m\mathcal{U}_m)^{\omega^i}| \dfrac{|A_m^{\omega^i\kappa_0^i}|}{|A_m^{\omega^i}|} & \text{if } i \text{ is odd.} \end{cases}$$

(2) *Let $m_0 \geq 0$, $i \geq 2$ and j be integers satisfying $i-j \not\equiv 0 (\bmod\ p-1)$. There exists a constant $C(m_0,i,j)$ such that for any integer $m \geq C(m_0,i,j)$, we have*

$$|H^2_{\acute{e}t}(\mathrm{Spec}(\mathbb{Z}[\mu_{p^{m_0+1}},1/p]),\mathbb{Z}_p(i))^{\omega^j}| = \begin{cases} |(\mathcal{U}_m/\overline{\mathcal{C}}_m)^{\omega^{i-j}\kappa^i_{m_0}}| & \text{if } i \equiv j \pmod 2, \\ |(\mathcal{U}_m/\overline{\mathcal{G}}^{(m_0)}_{m,i} + I_m\mathcal{U}_m)^{\omega^{i-j}}| \dfrac{|A_m^{\omega^{i-j}\kappa^i_{m_0}}|}{|A_m^{\omega^{i-j}}|} \\ \qquad\qquad\qquad \text{if } i \not\equiv j \pmod 2. \end{cases}$$

If we assume the non-vanishingness of the p-adic L-function at an integer, we get the folloing corollary.

Corollary 2. (1) *Let $m_0 \geq 0$, $i \geq 2$ be integers satisfying $i \not\equiv 0,1 \pmod{p-1}$. Assume $L_p(i,\omega^{1-i}) \neq 0$. There exists an constant $C(i)$ such that for any integer $m \geq C(i)$, we have*

$$|H^2_{\acute{e}t}(\mathrm{Spec}(\mathbb{Z}[1/p],\mathbb{Z}_p(i))| \sim_p \begin{cases} L_p(1-i,\omega^i) \sim_p B_i/i & \text{if } i \text{ is even}, \\ |(\mathcal{U}_m/\overline{\mathcal{G}}^{(0)}_{m,i} + I_m\mathcal{U}_m)^{\omega^i}| \dfrac{L_p(i,\omega^{1-i})}{\prod_{\psi\in\widehat{\Gamma}_m} B_{1,\omega^{-i}\psi}} & \text{if } i \text{ is odd}. \end{cases}$$

(2) *Let $m_0 \geq 0$, $i \geq 2$, j be integers satisfying $i-j \not\equiv 0,1 \pmod{p-1}$. Assume $L_p(i,\omega^{j-i+1}\psi) \neq 0$ for all $\psi \in \widehat{\Gamma}_{m_0}$. There exists an constant $C(m_0,i,j)$ such that for any integer $m \geq C(m_0,i,j)$, we have*

$$|H^2_{\acute{e}t}(\mathrm{Spec}(\mathbb{Z}[\mu_{p^{m_0+1}},1/p]),\mathbb{Z}_p(i))^{\omega^j}| \sim_p \begin{cases} \displaystyle\prod_{\psi\in\widehat{\Gamma}_{m_0}} L_p(1-i,\omega^{i-j}\psi) \sim_p \prod_{\psi\in\widehat{\Gamma}_{m_0}} (B_{i,\omega^{-j}\psi}/i) \\ \qquad\qquad\qquad \text{if } i \equiv j \pmod 2, \\ |(\mathcal{U}_m/\overline{\mathcal{G}}^{(m_0)}_{m,i} + I_m\mathcal{U}_m)^{\omega^{i-j}}| \dfrac{\prod_{\psi\in\widehat{\Gamma}_{m_0}} L_p(i,\omega^{j-i+1}\psi)}{\prod_{\psi\in\widehat{\Gamma}_m} B_{1,\omega^{j-i}\psi}} \\ \qquad\qquad\qquad \text{if } i \not\equiv j \pmod 2. \end{cases}$$

At the end of this article, we state the relation between Corollary 1, 2 and Conjecture 4. Under Conjecture 5 ($\mathrm{C}_{m,1-i}$) and Conjecture 3, we get the following equivalences.

Conjecture 4 for a pair (m, i)

$$\iff \quad |(\mathcal{U}_m/\overline{\mathcal{G}}_{m,i}^{(m_0)} + I_m\mathcal{U}_m)^{\omega^{i-j}}|\frac{|A_m^{\omega^{i-j}\kappa_{m_0}^i}|}{|A_m^{\omega^{i-j}}|} \sim_p 1$$

for any integer j such that $i \not\equiv j \pmod 2$.

$$\iff \quad |(\mathcal{U}_m/\overline{\mathcal{G}}_{m,i}^{(m_0)} + I_m\mathcal{U}_m)^{\omega^{i-j}}|\frac{\prod_{\psi\in\widehat{\Gamma}_{m_0}} L_p(i, \omega^{1-i+j}\psi)}{\prod_{\psi\in\widehat{\Gamma}_m} B_{1,\omega^{j-i}\psi}} \sim_p 1$$

for any integer j such that $i \not\equiv j \pmod 2$.

References

[1] M. Aoki, On the K-groups of algebraic integers of cyclotomic fields, preprint.

[2] A. Borel, Cohomologie réele stable des groupes S-arithmétiques classiques, C.R. Acad. Sci. Paris **271** (1970), 1156-1158.

[3] W. Dwyer and E.Friedlander, Algebraic and étale K-theory, Trans.AMS **292** (1985), 247-280.

[4] P. Elbaz-Vincent, H.Gangle and C.Soulé, Quelques calculs de la cohomologie de $GL_N(\mathbb{Z})$ et de la K-théorie de $\mathbb{Z}$, C.R.Acad.Sci.Paris, Ser.I **335** (2002), 321-324.

[5] K. Iwasawa, On some modules in the theory of cyclotomic fields, J. Math. Soc. Japan **16** (1964), 42-82.

[6] B. Kahn, "On the Lichtenbaum-Quillen conjecture" in Algebraic K-theory and Algebraic Topology, Nato Proceedings Lake Louise **407**, Kluwer, Dordrecht, (1993),147-166.

[7] M. Kolster, T.Nguyen Quang Do and V.Fleckinger, Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures, Duke Math.J. **84** (1996), 679-717.

[8] M. Kurihara, Some remarks on conjectures about cyclotomic fields and K-groups of Z, Compositio Math. **81** (1992), 223-236.

[9] R. Lee and R.H.Szczarba, The group $K_3(\mathbb{Z})$ is cyclic of order 48, Annals of Math. **104** (1976), 31-60.

[10] B. Mazur and A.Wiles, Class fields of abelian extensions of $\mathbb{Q}$, Invent. Math. **76** (1984), 179-330.

[11] J. Milnor, Introduction to Algebraic K-theory, Annals of Math. Study vol.72, Princeton University Press, (1971).

[12] S. Mitchell, On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic view point, Math. Sci. Res. Inst. Publ. vol.27, Algebraic topology and its applications, Springer, (1994) , 163-240.

[13] T. Nguyen Quang Do, Sur la $\mathbb{Z}_p$-torsion de certains modules galoisiens. Ann. Inst. Fourier **36** (1986), 27-46.

[14] D. Quillen, Finite generation of the groups K_i of rings of algebraic integers, Lecture Notes in Math. vol.341, Springer Verlag, (1973), 179-198.

[15] J. Rognes, $K_4(\mathbb{Z})$ is the trivial group, Topology **39** (2000), 267-281.

[16] P. Schneider, Über gewisse Galoiscohomologiegruppen, Math. Z. **168** (1979), 181-205.
[17] C. Soulé, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale, Invent. Math. **55** (1979), 251-295.
[18] L. Washington, Introduction to Cyclotomic Fields, 2nd ed., Graduate Texts in Mathematics, Vol.83, Springer-Verlag, Berlin/New York, (1997).

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo, Japan
E-mail address: `m-aoki@math.titech.ac.jp`