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ON THE FUNDAMENTAL GROUPS WITH RESTRICTED RAMIFICATION

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1. Introduction

In this Note I will explain the result of [3] whose subject is the arithmetic fundamental group with restricted ramification. The aim of this article is following:

⋆ We define a certain quotient of the étale fundamental group of a scheme which classifies étale coverings with bounded ramification along the boundary.
⋆ We show the finiteness of the abelianization of this group for an arithmetic scheme over a ring of integers of a number field.

Before going to the definition, we would like to explain importance of the finiteness of the fundamental groups in arithmetic theory: Let K be a finite extension over $\mathbb{Q}$ and $\widetilde{K}$ the narrow Hilbert class field over K, *i.e.*, the maximal unramified abelian extension over K and it admits ramification at the infinite places. The Galois group of $\widetilde{K}/K$ coincides with abelianized fundamental group as follows:

$$\pi_1(\operatorname{Spec}\mathcal{O}_K)^{\mathrm{ab}} = \operatorname{Gal}(\widetilde{K}/K).$$

By the classical class field theory, the Galois group is canonically isomorphic to the ideal class group of K and one can obtain the finiteness of the abelian fundamental group. Various attempts were made to generalize this to varieties of higher dimension, and as N. M. Katz and S. Lang showed in [4], the abelian étale fundamental group is finite for an arithmetic scheme.

On the other hand, the class field theory of Takagi-Artin admits ramification. Let K_D be a narrow ray class field associated with integral ideal $D \subset \mathcal{O}_K$. Using the fundamental group with restricted ramification mentioned above, we have the following description of the Galois group of K_D/K:

$$\pi_1(\operatorname{Spec}\mathcal{O}_K, D)^{\mathrm{ab}} = \operatorname{Gal}(K_D/K).$$

From class field theory, we have the finiteness of this abelian fundamental group with restricted ramification like the case of an étale fundamental group. The main result of [3] is the finiteness of the abelianization of this group for an arithmetic scheme.

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2. Definition

Abbes-Saito ramification filtration. Let K be a complete discrete valuation field, and let G_K be the absolute Galois group of K. By using techniques of rigid geometry, A. Abbes and T. Saito ([1]) defined a decreasing filtration $(G_K^a)_{a\in\mathbb{Q}_{\geq 0}}$ by closed normal subgroups G_K^a of G_K.

The filtration coincides with the classical upper numbering ramification filtration shifted by one, if the residue field of K is perfect (see [6], §IV.3 for the classical case). It is noteworthy that the filtration is left continuous and their jumps are rational. For a real number $a > 0$, we define G_K^{a+} to be the topological closure of $\cup_{b>a} G_K^b$, and put $G_K^{a-} = \cap_{b<a} G_K^b$, where b denotes a rational number. Then $G_K^{a-} = G_K^a$ if a is rational, and $G_K^{a-} = G_K^{a+}$ if a is not rational. In particular, G_K^1 is the inertia subgroup of G_K, and G_K^{1+} is the wild inertia subgroup of G_K. For any Galois extension L over K, the ramification filtration of the Galois group $\mathrm{Gal}(L/K)$ can be defined by $G_K^a/(G_K^a \cap G_L)$.

Fundamental Groups with Restricted Ramification. Let X be a connected normal noetherian scheme, $k(X)$ the function field of X, $\overline{k(X)}$ the separable closure of $k(X)$ and $D = \sum_i a_i \mathit{\Gamma}_i$ a Weil divisor of X with coefficients in $\mathcal{Q} := \{a, a+ \mid a \in \mathbb{Q}_{\geq 1}\}$.

Let $\pi_1(X, D)$ denote the quotient of $\pi_1(X \smallsetminus D)$ by the normal closed subgroup generated by the images of the a_i-th ramification subgroup $\mathrm{Gal}\left(\overline{k(X)}_{\overline{\xi}_i}/k(X)_{\xi_i}\right)^{a_i}$ for all generic point ξ_i of $\mathit{\Gamma}_i$ and $\overline{\xi}_i \mid \xi$. Here, the extension $\overline{k(X)}_{\overline{\xi_i}}/k(X)_{\xi_i}$ is the extension of complete discrete valuation field obtained by the completion at ξ_i and a prime $\overline{\xi}_i$ over ξ_i.

The group $\pi_1(X, D)$ is the fundamental group associated with the Galois category of the coverings with ramification bounded by D defined as follows: Let Y be a normal scheme over X, and we assume it is connected *for simplicity*. The morphism $Y \to X$ is said to be of *ramification bounded by* D, if it is finite étale over $X \smallsetminus D$ and, for each generic point ξ_i of $\mathit{\Gamma}_i$, we have $\mathrm{Gal}\left(k(Y)_{\eta_i}/k(X)_{\xi_i}\right)^{a_i} = 1$ for all $\eta_i \mid \xi_i$. The category of coverings of X which have ramification bounded by D forms a Galois category ([3], Theorem 2.5).

Relation to étale covers. By the Galois category theory ([2], §6), it is easy to see that we have the following surjection

$$\pi_1(X \smallsetminus D) \to \pi_1(X, D) \to \pi_1(X).$$

If the scheme X is regular and the Weil divisor $D = \sum a_i \mathit{\Gamma}_i$ with $a_i = 1$ for all i, then $\pi_1(X, D) = \pi_1(X)$ by the theorem of Zariski-Nagata on the purity of the branch locus (*cf.* [2], Exposé X, Théorème 3.1) .

Relation to tame covers. For a general X, if we have $a_i = 1+$ for all i, then $\pi_1(X, D) = \pi_1^{\mathrm{tame}}(X, D)$ where $\pi_1^{\mathrm{tame}}(X, D)$ is the tame fundamental group defined in Exposé XIII of [2].

3. A Finiteness Result

Theorem ([3], Theorem 1.1). *Let k be a finite extension of $\mathbb{Q}$ and X a normal scheme of finite type and faithfully flat over the ring of integers $\mathcal{O}$ of k whose geometric generic fiber $X \otimes_{\mathcal{O}} \overline{k}$ is connected. Then, the abelianized fundamental group $\pi_1(X, D)^{\mathrm{ab}}$ is finite for any Weil divisor D of X. Equivalently, there exist only finite many isomorphic classes of abelian coverings of X whose ramification is bounded by D.*

If $\dim X = 1$, then $\pi_1(X, D)^{\mathrm{ab}}$ is isomorphic to the ray class group of the number field $k(X)$ with modulus D. In this case, the above theorem is deduced from the classical class field theory. Furthermore, it is well known in the following cases:

- The finiteness of the abelian étale fundamental group $\pi_1(X)^{\mathrm{ab}}$ was shown by N. M. Katz and S. Lang in [4].
- The abelian tame fundamental group $\pi_1^{\mathrm{tame}}(X, D)^{\mathrm{ab}}$ is finite by the theorem of A. Schmidt in [5].

To prove the theorem, we basically follow the proof of the Schmidt's theorem ([5], Theorem 3.1).

References

[1] A. Abbes and T. Saito, Ramification of local fields with imperfect residue fields, Amer. J. Math. **124** (2002), no. 5, 879–920.

[2] A. Grothendieck, Revêtements étales et groupe fondamental (SGA 1), Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1960–1961, Lecture Notes in Mathematics, Vol. 224.

[3] T. Hiranouchi, *Finiteness of abelian fundamental groups with restricted ramification*, C. R. Acad. Sci. Paris, Ser. I **341** (2005), 207–210.

[4] N. M. Katz and S. Lang, Finiteness theorems in geometric classfield theory, Enseign. Math. (2) **27** (1981), no. 3-4, 285–319 (1982), With an appendix by Kenneth A. Ribet.

[5] A. Schmidt, Tame coverings of arithmetic schemes, Math. Ann. **322** (2002), no. 1, 1–18.

[6] J.-P. Serre, Corps locaux, Deuxième édition, Publications de l'Université de Nancago, No. VIII, Hermann, Paris, 1968.

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